PID Stabilization of a Position-Controlled Manipulator with Wrist Sensor

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Abstract—In this paper we discuss PID stabilization of a position-controlled manipulator equipped with a wrist sensor using Hermite-Biehler theorem framework. A second order transfer function for the manipulator is assumed along each coordinate direction that can be replaced by unity gain at slow speeds of manipulation. The wrist sensor impedance is similarly represented by a second order transfer function. Hermite-Biehler theorem is used for the stability analysis of the closed-loop system. The general theoretical framework presented here can be easily adapted to other low order plant models.

Index Terms—PID Control, Robot Control, Wrist Sensor

I. INTRODUCTION

Robot manipulators are used in a number of industrial and service applications [4-8]. In most cases robot can be considered as a positioning device that decouples the motion along each coordinate direction, and can be approximated with a second order transfer function. For example, Puma 560 robot has a natural frequency that is not less than 2 Hz [3, 9, 10]. In position and/or force control applications a high impedance wrist-sensor is normally used as end-effector, where the sensor can be similarly modeled with a second order transfer function. The wrist-sensor stiffness is usually high (of the order of 10^5 oz-in). In this paper we study the stability of the closed-loop system formed by the positioning manipulator, the wrist-sensor, and a PID controller. We use the Hermite-Biehler (HB) framework [1, 2, 11, 12] to analyze the stability of the closed-loop system. Hermite-Biehler and generalized HB theorems characterize the stability of a given polynomial and provide information on RHP root locations. In this paper we show how the characteristic polynomial of the robot-sensor plant can be cast in the HB framework. The paper then discusses synthesis of the PID controller for the problem. A general analysis-synthesis framework is developed that can be applied to similar plant models.

The organization of the paper is as follows: The problem is formulated in §2. In §3 the generalized Hermite-Biehler theorem and framework for stability determination of PID controlled plants are presented. In §4 the stability of the position-controlled manipulator with wrist-sensor is analyzed in HB framework, and necessary and sufficient conditions for stability are determined. An algorithm to select controller gains and an example are presented in §5.

II. PROBLEM FORMULATION

Consider a possibly unstable process given by the following transfer function

\[ G_p(s) = \frac{n_p}{d_p} = \frac{a s + b}{s^2 + c s + k} \]

where, the first part represents a sensor transfer function with known impedance and the second part represents a positive controlled manipulator. Assume that the process (1) is controlled through unity gain feedback by a PID controller whose transfer is given by

\[ G_c(s) = \frac{n_c}{d_c} = \frac{K_d s^2 + K_p s + K_i}{s} \]

Then, the closed-loop characteristic polynomial is

\[ \psi(s) = d_p(s) + (K_d + s^2 K_p + K_i) n_p(s) + s K_p n_c(s) \]

Define \( \tilde{A} \equiv \{a, b, c, k, n, m, \omega_n\} \subseteq \mathbb{R} \) to be the set of all system constants, and let \( \tilde{k} \equiv \{K_p, K_i, K_d\} \) represent the controller parameters; then the stability problem is: given \( \tilde{A} \), determine \( \tilde{k} \in \mathbb{R} \) \( \text{Re}[s] < 0 \) \( \forall s \in \tilde{A} \) \( \psi(s) = 0 \).

III. HERMITE-BIEHLER FRAMEWORK

Theorem 3.1 (Generalized Hermite-Biehler Theorem)

\[ \delta(s) = \sum_{i=0}^{n} \delta_i s_i, \quad \delta_i \in \mathbb{R} \quad \forall i \quad \text{with a root at the origin of multiplicity } k. \]

Writing \( \delta(s) = \delta_e(s^2) + s \delta_o(s^2) \) where, \( \delta_{e,o}(s^2) \) are the components of \( \delta(s) \) made up of even and odd powers of
s, respectively. For every \( \omega \in \mathbb{R} \), denote \( \delta_j(\omega) = p(\omega) + jq(\omega) \), where, \( p(\omega) = \delta_{\omega}(-\omega^2) \), \( q(\omega) = \omega \delta_{\omega}(-\omega^2) \) and let \( \omega_y \) denote the real non-negative distinct zeros of \( \delta_{\omega}(-\omega^2) \), and let \( \omega_w \) denote the real non-negative distinct zeros of \( \delta_{\omega}(-\omega^2) \), both arranged in ascending order of magnitude. Denote \( 0 < \omega_{a_1} < \omega_{a_2} < \ldots < \omega_{a_m} \) to be the zeros of \( q(\omega) \) that are real, distinct and nonnegative. Also, define \( \omega_o = 0 \), \( \omega_{om} = \infty \) and \( p^{(k)}(\omega_o) = \left( \frac{d^k}{d\omega^k} p(\omega) \right)_{\omega=\omega_m} \). Then,

\[
\sigma[\delta(s)] = (-1)^{m-1} + 2 \sum_{j=1}^{m-1} \text{sgn}[p(\omega_{a_j})] \text{sgn}[q(\omega_{a_j})] 
\]

(5)

\[
\sigma[\delta(s)] = (-1)^{m-1} + 2 \sum_{j=1}^{m-1} \text{sgn}[p(\omega_{a_j})] \text{sgn}[q(\omega_{a_j})] 
\]

(6)

Assuming a feasible set of \( \vec{I}_i \) \( \forall i \in \{0, m_q \} \) exists, we denote this set as \( \theta^{*} \). Then, a non-empty stabilizing PID set \( S_\varepsilon \equiv \{K_p, K_i, K_d, \vec{K}_1, \vec{K}_d, \vec{K} \} \) for the problem can be computed from a set of linear inequalities given by \( \text{sgn} \left( \left[ P_1 \right] \left[ P_2 \right] \left[ \varepsilon \right] \right) = 0 \), where the matrices \( P_1 \in \mathbb{R}^{n \times d}, P_2 \in \mathbb{R}^{m \times d} \), and \( \varepsilon \in \mathbb{R}^{d \times d} \) are defined as

\[
\left[ P_1 \right] = \left[ p_1(\omega_{a_1}) \quad p_1(\omega_{a_2}) \quad \ldots \quad p_1(\omega_{a_m}) \right]^T 
\]

(7)

\[
\left[ P_2 \right] = \left[ p_2(\omega_{a_1}) \quad p_2(\omega_{a_2}) \quad \ldots \quad p_2(\omega_{a_m}) \right]^T 
\]

(8)

\[
\left[ \varepsilon \right] = \left[ K_i \quad K_d \right]^T 
\]

(9)

Note that in the preceding analysis \( S_\varepsilon \in \mathbb{R} \) has been assumed. In practice, to facilitate tuning, we may require that \( S_\varepsilon \in \mathbb{R}^+ \), i.e., \( \{S_\varepsilon \in \mathbb{R}^+: \left[ P_1 \right] + \left[ P_2 \right] \left[ \varepsilon \right] = 0 \} \neq \emptyset \). \( S_\varepsilon \in \mathbb{R}^+ \).

IV. STABILITY ANALYSIS OF ROBOT MANIPULATOR

A. Necessary Conditions for Stability

First, we shall determine the necessary condition for the existence of \( \vec{K} \). To this effect, re-write (1) as

\[
G_p(s) = \frac{n_p(s)}{d_p(s)} = \frac{a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}{s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5} 
\]

(10)
where, \( \{a_i, b_i, f_i\} \in \mathbb{R}^1 \), \( i, f \). Note that for the given problem \( \Theta[(y, s)] = m_y = 5, \rho = 4, \alpha = 3 \), and the polynomial

\[
q(\omega) = a_0^2 + \omega^2 \left( a_0 b_0 - b_0^2 - b_2 a_0 + a_2 a_0^2 \right)
\]

(11)

Let \( y = (-1)^n b_0 a_0 + a_2 a_0^2 \omega^2 \) be the number of roots of \( q(\omega, K_p) \) that are real, nonnegative, and distinct, given as: \( \omega_0 = 0 \), and the four roots of the equation

\[
\tilde{q}(\omega, K_p) = 0.
\]

Let \( \Delta \) be the logical set defined as

\[
\Delta = \{ \omega \} \land \{ a_i \} \land \{ b_i \} = -1 \land \sum_i \text{sgn}(\lambda_i) \geq 0 \forall i \in \{a_i, b_i\};
\]

then, \( \omega_1, \omega_2, \omega_3, \), and \( \omega_4 \) are the roots of \( \tilde{q}(\omega, K_p) = 0 \). Let \( A \) denote logical AND, \( \lor \) denotes logical OR, \( \mathbb{R}^+ = (0, \infty), \mathbb{R}^- = (0, \infty), [\mathbb{R} \neq 0] = \mathbb{R}^+ \cup \mathbb{R}^- \), and \( C \) denotes the set of complex numbers. Let \( 0 \leq m^* \leq 4 \) be the number of roots of \( \tilde{q}(\omega, K_p) = 0 \) that are real, nonnegative, and distinct. Table 1 summarizes the results for the various cases of \( m^* \).

Table 1: Stability results for various cases of roots of \( \tilde{q}(\omega, K_p) \)

<table>
<thead>
<tr>
<th>( m^* )</th>
<th>Possible ( {\omega_i} ) for ( g^* \neq \emptyset )</th>
<th>Condition on sign of ( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \emptyset )</td>
<td>Not applicable</td>
</tr>
<tr>
<td>1</td>
<td>( 2 \mathbb{R}, 2 \mathbb{C} )</td>
<td>( \text{sgn}(\Delta) \neq -1 \land \sum_i \text{sgn}(\lambda_i) = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 4 \mathbb{R} \neq 0 ), ( 0, 0, 2 \mathbb{R} )</td>
<td>( \text{sgn}(\Delta) \neq -1 \land \sum_i \text{sgn}(\lambda_i) \geq 0 )</td>
</tr>
<tr>
<td>3, 4</td>
<td>( \emptyset )</td>
<td>Not applicable</td>
</tr>
</tbody>
</table>

The necessary condition for stability of the closed-loop system defined by (1) and (2) is given by the following lemma.

**Lemma 4.1:** Let \( \Omega \) be a logical set defined as

\[
\Omega \equiv \text{sgn}(\Delta) \neq -1 \land \sum_i \text{sgn}(\lambda_i) \geq 0 \forall i \in \{a_i, b_i\};
\]

then, the necessary condition for a set of stabilizing PID controllers for the process given by (1) is

\[
\Omega \neq \emptyset.
\]

**Proof:** From table 1 the conditions for \( g^* \neq \emptyset \) are

\[
\begin{align*}
1 & \iff \text{sgn}(\Delta) \neq -1 \land \sum_i \text{sgn}(\lambda_i) = 1 \lor [a_i, b_i] \\
2 & \iff \text{sgn}(\Delta) \neq -1 \land \sum_i \text{sgn}(\lambda_i) \leq 0 \lor [a_i, b_i]
\end{align*}
\]

Combining the two conditions and recognizing that

\[
\prod_i \text{sgn}(\lambda_i) = -1 \land \prod_i \text{sgn}(\lambda_i) \geq 0 \Rightarrow \sum_i \text{sgn}(\lambda_i) \geq 0,
\]

we get the result \( g^* \neq \emptyset \iff \Omega \neq \emptyset. \)

**Lemma 4.2:** \( \text{sgn}(\Delta) \neq -1 \) if and only if \( K_p \in S^*(K_p) \)

where, \( \Delta \) is as defined above, and

\[
S^*(K_p) \equiv \left( -\infty, \text{min}\{k_{1,2}^*, \} \right) \cup \left( \text{max}\{k_{1,2}^* \}, \infty \right), k_{1,2}^* \in \mathbb{R}
\]

where, \( k_{1,2}^* \) are the roots of \( \Delta \).

**Proof:** We note that we can write \( \Delta \) as a quadratic in \( K_p \), \( \Delta = aK_p^2 + bK_p + c = \prod_i (K_p - k_i^*), i = 1, 2, \)

Then, it can be seen that

\[
\Delta \geq 0 \forall K_p \in \left( -\infty, \text{min}\{k_{1,2}^* \} \right) \cup \left( \text{max}\{k_{1,2}^* \}, \infty \right)
\]

and

\[
\Delta < 0 \forall K_p \in \left( \text{min}\{k_{1,2}^* \}, \text{max}\{k_{1,2}^* \} \right). \]

Therefore, \( \text{sgn}(\Delta) \neq -1 \land \forall K_p \in \left( -\infty, \text{min}\{k_{1,2}^* \} \right) \cup \left( \text{max}\{k_{1,2}^* \}, \infty \right) \equiv S^*(K_p) \lor [a], \{b\}
\]

However, it can be verified that for \( \forall k_{1,2}^* \in \mathbb{C}, \)

\[
\Delta > 0 \forall K_p \Rightarrow S^*(K_p) = (-\infty, \infty).
\]

\[\square\]
Lemma 4.3: For \( \sum_{i} \text{sgn}[\Delta_i] \geq 0 \) the range of \( K_p \) is given as \( \{K_p \in \mathcal{S}(K_p) \} \) where, \( \mathcal{S}(K_p) \equiv \left\{ \left( -\infty, \frac{b_1 a_0 - b_0 a_1 - K_p a^2}{a_1^2} \right) \right\} \) and \( \text{sgn} \left[ \frac{b_1 a_0 - b_0 a_1 - K_p a^2}{a_1^2} \right] \equiv s_i \).

Proof: Note that

\[ \lambda_{1,2} = \frac{b_1 a_0 - b_0 a_1 - K_p a^2 \pm \sqrt{\Delta}}{2(a_0 - b_3 a_1)} = \frac{b(K_p) \pm \sqrt{\Delta}}{A(k, b)} \quad (14) \]

Then, regardless of the value of \( \text{sgn}[-1] \), since \( \Delta \geq 0 \) \( \forall K \in \mathcal{S}(K_p) \),

\[ \sum_{i} \text{sgn}[\Delta_i] \geq 0 \equiv \text{sgn}[\Delta] \equiv 1 \wedge \text{sgn}[\Delta_2] \equiv -1 \]

But \( \text{sgn}[\Delta] = 1 \iff K_p \in \mathcal{S}(K_p) \), while \( \text{sgn}[\Delta] = -1 \iff K_p \in \mathcal{S}(K_p) \).

\[ \text{sgn}[\Delta_2] \equiv -1 \equiv \text{sgn}[\Delta^2_2 - \Delta_2^2] \]

Since \( s_i \) is independent of \( K_p \)

\[ \{K_p, \text{sgn}[-1], \text{sgn}[\Delta_2] \} = \left\{ (-\infty, \infty) \right\} \text{ or } \{K_p, \text{sgn}[-1], \text{sgn}[\Delta_2] \} = \{K_p, \text{sgn}[-1], \text{sgn}[\Delta_2] \} = \{0\} \text{ for } s_i = -1 \]

\[ \Rightarrow \mathcal{S}(K_p) \text{ for } s_i = -1 \]

\[ \Rightarrow \{0\} \text{ for } s_i = 1 \]

Theorem 1: The process given by (1) is stabilizable i.e., \( \exists \left[ g^* \right] \neq \varnothing \) if and only if \( K_p \in \mathcal{S}(K_p) \) where, \( \mathcal{S}(K_p) \) is evaluated as

(i) \( \mathcal{S}(K_p) \equiv \left\{ \mathcal{S}^*(K_p) \text{ for } s_i = -1 \wedge \mathcal{S}^*(K_p) \text{ for } s_i = 1 \right\} \) \( \forall k_{1,2} \in \mathbb{R} \)

(ii) \( \mathcal{S}(K_p) \equiv \left\{ \mathcal{S}^*(K_p) \text{ for } s_i = -1 \wedge \mathcal{S}^*(K_p) \text{ for } s_i = 1 \right\} \) \( \forall k_{1,2} \in \mathbb{C} \) where, \( \mathcal{S}^*(K_p) \) and \( \mathcal{S}(K_p) \) are as defined in lemmas 4.2 and 4.3.

Proof: (i) From lemmas 4.2 and 4.3 we get

\[ \text{sgn}[\Delta] \equiv -1 \wedge \sum_{i} \text{sgn}[\Delta_i] \equiv 0 \equiv K_p \]

\[ \in \mathcal{S}^*(K_p) \cap \mathcal{S}(K_p) \]

Substituting with the result of lemma 4.1, we have

\[ \left\{ \mathcal{S}^*(K_p) \cap \mathcal{S}(K_p) \right\} \text{ for } s_i = -1 \]

\[ \Rightarrow \mathcal{S}(K_p) = \left\{ \mathcal{S}^*(K_p) \cap \mathcal{S}(K_p) \right\} \text{ for } s_i = 1 \]

Since \( \mathcal{S}(K_p) \cap \mathcal{S}(K_p) = (-\infty, \infty) \) and \( \mathcal{S}^*(K_p) \cap (-\infty, \infty) = \mathcal{S}^*(K_p) \) we obtain

\[ \mathcal{S}(K_p) = \left\{ \mathcal{S}^*(K_p) \right\} \text{ for } s_i = -1 \]

\[ \mathcal{S}^*(K_p) \cap \mathcal{S}(K_p) \text{ for } s_i = 1 \]

(ii) Observe from lemma 4.2 that \( \forall k_{1,2} \in \mathbb{C} \), \( \mathcal{S}^*(K_p) = (-\infty, \infty) \). Using this result in (i), and recognizing that \( (-\infty, \infty) \cap \mathcal{S}(K_p) = \mathcal{S}(K_p) \), the result follows.

B. Sufficient Conditions for Stability

We would now like to determine sufficient conditions for the existence of the stabilizing PID set in terms of the system parameters. From [11, 12] we note that the sufficient condition for the PID to exist is given by

\[ \left[ g^* \right] \neq \varnothing \Rightarrow \exists g^* \]

where, the matrices \( \left[ R_{1,1} \right], \left[ R_{2,1} \right], \) and \( \left[ R_{2,2} \right] \) are defined by (5)-(7), and from table 1 we know that \( g^* \neq \varnothing \neq g^* \). Define

\[ f \equiv \left\{ \begin{array}{l} a_1 \max(\lambda_1)^3 + (a_0 b_3 - a_1 b_2) \max(\lambda_1)^2 \\
+ (K_p a^2 + a_0 b_2) \max(\lambda_1) + K_p a^2 \\
+ \max(\lambda_1) (a^2 + \lambda_1) \end{array} \right\} \]

\[ \eta_{1,2} \equiv \frac{a_1 \max(\lambda_1)^3 + (a_0 b_3 - a_1 b_2) \max(\lambda_1)^2 \\
+ (K_p a^2 + a_0 b_2) \max(\lambda_1) + K_p a^2 \\
+ \max(\lambda_1) (a^2 + \lambda_1)}{\lambda_{1,2} (a^2 + \lambda_{1,2} + a_0^2)} \]

\[ \eta_{1,2} \equiv \frac{a_1 \max(\lambda_1)^3 + (a_0 b_3 - a_1 b_2) \max(\lambda_1)^2 \\
+ (K_p a^2 + a_0 b_2) \max(\lambda_1) + K_p a^2 \\
+ \max(\lambda_1) (a^2 + \lambda_{1,2})}{\lambda_{1,2} (a^2 + \lambda_{1,2} + a_0^2)} \], and
Lemma 4.4: (i) If $\tilde{m}^* = 1$, then the sufficient condition for the existence of the stabilizing PID set $\tilde{K}$, for the process whose transfer function is given by (10), is given by:

$$
\begin{align*}
K_i &\in (0, \infty), K_d \in (f, \infty) \forall \gamma = 1 \\
K_i &\in (-\infty, 0), K_d \in (-\infty, f) \forall \gamma = -1
\end{align*}
$$

\forall K_p \in S(K_p)$ where, $\lambda$ and $\gamma$ are as defined before, and $S(K_p)$ is given by theorem 1.

(ii) If $\tilde{m}^* = 2$, then the set of sufficient conditions for the existence of the stabilizing PID set $\tilde{K}$, for the process whose transfer function is given by (12), are given as: $\forall K_p \in S(K_p)$

(a) if $\sum \sgn[\lambda_i] = 1$, then

$$
\begin{align*}
K_i &\in (0, \infty), K_d \in (f, \infty) \forall \gamma = 1 \\
K_i &\in (-\infty, 0), K_d \in (-\infty, f) \forall \gamma = -1
\end{align*}
$$

(b) if $\sum \sgn[\lambda_i] = 2$, then

$$
\begin{align*}
K_i &\in \mathbb{R}^+, K_d \in S_\eta \\
K_i &\in \mathbb{R}^-, K_d \in S_{\overline{\eta}}
\end{align*}
$$

Proof: (i) It is obvious from table 1 that whenever $\tilde{m}^* = 1$, $\sgn(A) \neq -1 \land \prod \sgn[\lambda_i] = -1 \Rightarrow \sgn[\Delta] \neq -1 \land \sum \sgn[\lambda_i] = 0$.

But from theorem 1 and lemma 4.1 we have $\sgn[\Delta] \neq -1 \land \sum \sgn[\lambda_i] \geq 0 \ \forall K_p \in S(K_p)$. Thus.

whenever $\tilde{m}^* = 1$, $\sum \sgn[\lambda_i] = 0 \ \forall K_p \in S(K_p)$.

Using this in (13) we deduce $\omega_{ol} = \sqrt{\max(\lambda_1)}$.

Furthermore, from (5)-(7), and the set of linear inequalities we obtain

$$
\begin{align*}
K_i > 0, K_d > \left[ p_1(\omega_{ol}) + |K|_2 p_2(\omega_{ol}) \right] \max(\lambda_i) p_2(\omega_{ol}) \forall \gamma = 1 \\
K_i < 0, K_d < \left[ p_1(\omega_{ol}) - |K|_2 p_2(\omega_{ol}) \right] \max(\lambda_i) p_2(\omega_{ol}) \forall \gamma = -1
\end{align*}
$$

Substituting for $\omega_{ol}$ in the expressions of $p_{1,2}(\omega)$ (see [11] for details) we obtain

$$
\begin{align*}
p_1(\omega_{ol}) + |K|_2 p_2(\omega_{ol}) &= f, \\
p_1(\omega_{ol}) - |K|_2 p_2(\omega_{ol}) &= \bar{f}
\end{align*}
$$

(ii-a) If $\sum \sgn[\lambda_i] = 1$, then $\{\omega_i\} = \{0, 0, 2 \Re \neq 0\}$. Then the root distribution of $q(\omega, K_p)$ is $\{0, 0, 2 \Re \neq 0\}$.

But since $\tilde{m}_q$ by definition, is the number of real, nonnegative, and distinct roots, therefore the roots that qualify for the computation of the LMI given by (15) are $\{\omega_0 = 0, \omega_{ol} = \sqrt{\max(\lambda_1)}\}$, and the proof follows from lemma 4.4.

(ii-b) The proof is on similar lines as (ii-a) except that for $\sum \sgn[\lambda_i] = 2$, the dimensions of (5) and (6) increase by one. It is obvious from (13) that for $\sum \sgn[\lambda_i] = 2$, $\omega_{ol,2} = \sqrt{\bar{\lambda}_{1,2}}$. The result follows by case (ii-a) in conjunction with the LMI given by (15), and making the substitution for $\omega_{ol,2}$.

Theorem 2 (Main result on stabilization): The process given by (10) is stabilizable with a PID controller if and only if $K_p \in S(K_p)$ and if the ranges of $K_i$, $K_d$ are chosen according to the following situations:

$$
\forall K_p \in S(K_p)
$$

(a) $\sum \sgn[\lambda_i] = 0, 1$ and $\gamma = 1$ then,

$$
K_i \in \mathbb{R}^+, K_d \in (f, \infty)
$$

(b) $\sum \sgn[\lambda_i] = 0, 1$ and $\gamma = -1$ then,

$$
K_i \in \mathbb{R}^-, K_d \in (-\infty, f)
$$

(c) $\sum \sgn[\lambda_i] = 2$ and $\gamma = 1$ then, $K_i \in \mathbb{R}^+, K_d \in S_\eta$ or $K_i \in \mathbb{R}^-, K_d \in S_{\overline{\eta}}$ whichever is non-empty.

(d) $\sum \sgn[\lambda_i] = 2$ and $\gamma = -1$ then,

$$
K_i \in \mathbb{R}^+, K_d \in S_\eta \lor K_i \in \mathbb{R}^-, K_d \in S_{\overline{\eta}} \lor K_i \in \mathbb{R}^-, K_d \in S_{\overline{\eta}} \lor K_i \in \mathbb{R}^+, K_d \in S_\eta \lor K_i \in \mathbb{R}^-, K_d \in S_{\overline{\eta}} \lor K_i \in \mathbb{R}^+, K_d \in S_\eta \lor K_i \in \mathbb{R}^-, K_d \in S_{\overline{\eta}}$$

Theorem 2 (Main result on stabilization): The process given by (10) is stabilizable with a PID controller if and only if $K_p \in S(K_p)$ and if the ranges of $K_i$, $K_d$ are chosen according to the following situations:

The following algorithm can be used to determine the stabilizing PID set for the given problem:

(i) Use the theorem 1 to calculate $S(K_p)$. Constraint the value of $K_p$ to $K_p = K_p \in S(K_p)$. 

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(ii) Calculate the value of \( \sum \text{sgn}(\lambda_i) \) and \( \gamma \). Note that \( \gamma \) is given as \( \gamma = (-1)^{\sum \text{sgn}(\lambda_i)} \)

Accordingly, calculate \( f \) (or \( \tilde{f} \)) or \( S_{\eta} \) (or \( S_{\eta} \)).

(iii) Use theorem 2 to evaluate the stabilizing PID set.

V. EXAMPLE

Consider a fourth-order process whose transfer function is of the form given by (1) with a process parameter set

\[ \mathcal{A} \equiv \{a, b, k, c, m, \xi, \omega_a\} = \{0.03, 10.25, 101, 10, 1, 0.25, 2\} \]

controlled with a PID controller whose transfer function is of a form given by (2). From \( \mathcal{A} \), we have

\[ \{a\} = \{a_1, a_2\} = \{-4000, -400\} \]

\[ \{b\} = \{b_1, b_2, b_3, b_4\} = \{11.15, 141, 404\} \]

Then, \( S_1 = -1 \), \( \Delta = 10^{11} \cdot 2560 K_p^2 - 165.76 K_p + 2.68 \rightarrow k_i^* \in \mathcal{C} \). Using lemma 4.2, \( S'(K,0) = (-\infty, \infty) \). Furthermore, using theorem 1, we find that \( S(K_p) = S'(K,0) = (-\infty, \infty) \). Let \( K_p = K_{p0} = 1 \). Then, \( \Delta = 2.3969 \times 10^{14} \). Substituting in (13) we obtain

\[ \lambda_1 = -355.09, \lambda_2 = 1.04 \times 10^{-4} \Rightarrow \sum \text{sgn}(\lambda_i) = 0 \]

\[ \max(\lambda_i) = 1.04 \times 10^{-4} \]. Thus,

\[ m_0 = 2 \Rightarrow \gamma = (-1)^{\sum \text{sgn}(\lambda_i)} \cdot 436000 = -1 \]. Using the value of \( \gamma \) and \( \sum \text{sgn}(\lambda_i) \) in theorem 2 we obtain

\[ K_p \in \mathcal{K} = (-\infty, 0) \text{ and } K_{p0} \in (-\infty, \tilde{f}) \text{ where, the value of } \tilde{f} \text{ depends on the constrained value of } K_p. \]

Thus for \( K_p = K_{p0} = 1 \), the stabilizing PID set is given by

\[ K = \left\{ (\infty, 0), (\infty, \tilde{f}) \right\} \]. Furthermore, constraining the value of \( K_i \), we obtain \( \tilde{f} = -9615.03 \) or, from lemma 4.4, \( \tilde{K} = \{(-\infty, 0), (-\infty, -9615.03)\} \). Similarly, the process can be repeated to get a family of stabilizing sets \( \tilde{K} \) by sweeping over \( VK_i \in (\infty, 0) \) with a constrained \( \lambda = K_{I0} = 1 \). The closed-loop is simulated for \( \tilde{K} = \{(-\infty, 0), (-\infty, -9615.03)\} \) and is shown in Fig. 1.

VI. CONCLUSION

In conclusion, this paper discusses the PID stabilization of a robot manipulator equipped with a wrist sensor. The formulation of the problem has been presented in a general framework that can also be applied to other problems. Analytical results on stability have been derived using Hermite-Biehler theorem that describes the interlacing property of the even and odd parts of the characteristic polynomial. The resulting algorithm provides a convenient method for PID gain selection based on the system parameters.

VII. REFERENCES


