A Preliminary Control Design for the Nonlinear Benchmark Problem

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Abstract

A nonlinear control problem has been posed by Bupp, Bernstein, and Coppola [1] to provide a benchmark for evaluating various nonlinear control design techniques. In this paper, the state-dependent Riccati equation (SDRE) technique detailed in [2] is utilized to produce a preliminary design for the benchmark problem. Analysis of the design shows that in the absence of disturbances and uncertainties, the SDRE nonlinear feedback solution compares very favorably to the optimal open-loop solution obtained via numerical optimization. It is also shown that the closed-loop system has stability robustness against parametric variations and low-frequency disturbances and attenuates middle-frequency disturbances.

1. Introduction

Numerous design methodologies exist for the control design of highly nonlinear systems. These include any of a huge number of linear design techniques [3, 4, 5, 6] used in conjunction with gain scheduling [7, 8, 9]; nonlinear design methodologies such as dynamic inversion [10], sliding mode control [10], and recursive backstepping [11]; and adaptive techniques which encompass both linear adaptive [12] and nonlinear adaptive [11] control. A lesser known nonlinear design procedure is the state-dependent Riccati equation (SDRE) technique [2]. This technique was used in [13] to produce advanced guidance algorithms, used in [14] in a full information nonlinear minimum energy design, used in [15] in an output feedback nonlinear minimum energy with respect to white noise design and briefly mentioned in [16]. However, [2] is the first work that we are aware of that explores the SDRE method in detail, providing a deeper understanding of the technique, its capabilities, and limitations. A brief overview of the SDRE method and some of the key results obtained in [2] follow.

The SDRE Method for Complete State Information

Consider the general infinite-horizon nonlinear regulator problem of the form:

\[ I = \frac{1}{2} \int_{t_0}^{\infty} x^T Q(x)x + u^T R(x)u \, dt \] (1)

with respect to the state \( x \) and control \( u \) subject to the nonlinear differential constraint

\[ \dot{x} = f(x) + g(x)u \] (2)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( f(x) \in C^\infty \), \( g(x) \in C^\infty \), \( Q(x) \in C^\infty \), \( R(x) \in C^\infty \), and where \( Q(x) = C^T(x)C(x) \geq 0 \), and \( R(x) > 0 \) for all \( x \). Here it is assumed that \( f(0) = 0 \) and \( g(x) \neq 0 \) for all \( x \neq 0 \).

The SDRE approach for obtaining a suboptimal solution of Problem (1)-(2) is:

i) Use direct parameterization to bring the nonlinear dynamics to the state-dependent coefficient (SDC) form

\[ \dot{x} = A(x)x + B(x)u \] (3)

where

\[ f(x) = A(x)x \quad B(x) = g(x) \] (4)

ii) Solve the state-dependent Riccati equation

\[ A^T(x)P + PA(x) - PB(x)R^{-1}(x)B^T(x)P + Q(x) = 0 \] (5)

to obtain \( P \geq 0 \), where \( P \) is a function of \( x \).

iii) Construct the nonlinear feedback controller

\[ u = -R^{-1}(x)B^T(x)P(x)x \] (6)

Associated with the SDC form (3), we have the following definitions.

Definition. \( A(x) \) is a detectable parameterization.
of the nonlinear system [in a region $\Omega$] if the pair \{${C}(x), {A}(x)$\} is detectable for all $x \in \Omega$.

**Definition.** \( {A}(x) \) is a stabilizable parameterization of the nonlinear system [in a region $\Omega$] if the pair \{${A}(x), {B}(x)$\} is stabilizable for all $x \in \Omega$.

**Properties of the SDRE Method**

The properties of the SDRE method will be stated as theorems along with some discussion. The proofs are given in [2] and will be omitted here.

**Theorem 1.** In addition to $f(x)$, $B(x) = g(x)$, $Q(x)$, and $R(x)$ belonging to $C^\infty$, assume that the SDC parameterization $A(x)$ is smooth (i.e., $C^\infty$) and is both detectable and stabilizable. Then the SDRE method produces a closed loop solution which is locally asymptotically stable.

This theorem is based on the fact that under the assumptions, the closed loop state-dependent coefficient matrix $A(x) - B(x)R^{-1}(x)B^T(x)P(x)$ is smooth and can be expanded in a Taylor series about the origin to yield

$$\dot{x} = [A(0) - B(0)R^{-1}(0)B^T(0)P(0)]x + \psi(x) \cdot \|x\| \quad (7)$$

where

$$\lim_{\|x\| \to 0} \psi(x) = 0 \quad (8)$$

In a neighborhood about the origin, the linear term which has a constant stable coefficient matrix dominates the higher order term yielding local asymptotic stability.

**Theorem 2.** In the case of scalar $x$, the SDRE nonlinear feedback solution is globally optimal.

This theorem highlights one of the amazing properties of the SDRE method. Even when the performance index (1) is nonquadratic, i.e., $Q = Q(x)$ and $R = R(x)$, in the scalar case the method produces the optimal solution of the regulator problem (1)-(2) in feedback form.

**Theorem 3.** With $\lambda = P(x)x$ the necessary condition for optimality $H_x = 0$, where $H$ is the Hamiltonian of the system, is always satisfied.

**Theorem 4.** Assume that the functions $A(x), B(x) = g(x), P(x), Q(x),$ and $R(x)$, along with their gradients, $A_x(x), B_x(x), P_x(x), Q_x(x),$ and $R_x(x)$, are bounded along trajectories. Then, under stability, as the state $x$ is driven to zero, with $\lambda = P(x)x$ the necessary condition $H_x = 0$ is asymptotically satisfied at a quadratic rate.

This last result gives rise to the following phenomenon that can be observed in example problems: As the states are driven toward zero, the SDRE control trajectories converge to the optimal control trajectories, the latter being obtained iteratively numerically.

Finally, the SDRE method allows one to impose hard limits on the control, control rate, and even control acceleration. An example of how this can be achieved for bounded control is given in [2]. This capability of the method has not been utilized in the preliminary design, but will be used in the next design iteration.

The rest of the paper is organized as follows. In the next section, the nonlinear benchmark problem is stated. In Section 3, a nonlinear regulator problem is posed for application by the SDRE method. Design results are then presented in Section 4 and the paper is concluded with a summary section.

2. The Nonlinear Benchmark Problem

As presented by Bupp, Bernstein and Coppola [1] the benchmark problem is a translational oscillator with eccentric rotational proof-mass actuator as shown in Figure 1.

**Figure 1:** Translational Oscillator/Rotational Actuator

With $q$ and $\dot{q}$ denoting the translational position and velocity of the cart and $\theta$ and $\dot{\theta}$ denoting the angular position and velocity of the rotational proof-mass, the equations of motion are:

$$(M + m) \ddot{q} + kq = -m \dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta + F \quad (9)$$

$$(I + me^2) \ddot{\theta} = -me \dot{q} \cos \theta + N \quad (10)$$

With the normalization
The equations of motion become

\[ \ddot{\xi} + \dot{\xi} = \epsilon(\dot{\theta}^2 \sin \theta - \dot{\theta} \cos \theta) + w \]  

(13)

\[ \dot{\theta} = -\epsilon \dot{\xi} \cos \theta + u \]  

(14)

The parameter values of the system are given in the following table.

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cart Mass</td>
<td>M</td>
<td>1.3608</td>
<td>kg</td>
</tr>
<tr>
<td>Arm Mass</td>
<td>m</td>
<td>0.096</td>
<td>kg</td>
</tr>
<tr>
<td>Arm Eccentricity</td>
<td>e</td>
<td>0.0592</td>
<td>m</td>
</tr>
<tr>
<td>Arm Inertia</td>
<td>I</td>
<td>0.0002175</td>
<td>kg-m²</td>
</tr>
<tr>
<td>Spring Stiffness</td>
<td>k</td>
<td>186.3</td>
<td>N/m</td>
</tr>
<tr>
<td>Coupling Parameter</td>
<td>e</td>
<td>0.200</td>
<td>-</td>
</tr>
</tbody>
</table>

The physical configuration of the system necessitates the constraint

\[ q | \leq 0.025 \text{ m} \]  

(15)

In addition, the control torque is limited by

\[ N \leq 0.100 \text{ N-m} \]  

(16)

although somewhat higher torques can be tolerated for short periods. These constraints relate to a maximum nondimensional cart displacement of \( \xi = 1.28 \) and an approximate maximum control of \( u = 1.4 \). Defining \( x = [x_1, x_2, x_3, x_4]^T = [\xi, \dot{\xi}, \sin \theta, \dot{\theta} \cos \theta]^T \), it can be shown that the nondimensional equations of motion in first order form are given by

\[ \dot{x} = f(x) + g(x)u + d(x)w \]  

(17)

where \( f(x), g(x), \) and \( d(x) \) are given by

\[
\begin{bmatrix}
  x_2 \\
  -x_1 \frac{(1-x_3^2)+x_2x_3}{(1-x_3^2)\Delta} \\
  \frac{x_4}{(1-x_3^2)\Delta} \\
  \frac{x_3}{(1-x_3^2)\Delta}
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  -\epsilon \cos \theta \\
  \Delta \\
  \Delta
\end{bmatrix} \begin{bmatrix}
  0 \\
  \frac{1}{\Delta} \\
  0 \\
  \frac{-\epsilon(1-x_3^2)}{\Delta}
\end{bmatrix}
\]  

(18)

respectively with

\[ \Delta = 1 - \epsilon^2 + \epsilon^2 x_3^2 \]  

(19)

3. The Nonlinear Regulator Problem

Consider the problem of minimizing the nonlinear performance index

\[ J = \int_0^\infty q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + q_4 \frac{x_4^2}{1-x_3^2} + ru^2 \, dt \]  

(20)

subject to the constraint

\[ \dot{x} = f(x) + g(x)u \]  

(21)

where \( f(x) \) and \( g(x) \) are given in (18). Noting that \( x_3 = \sin \theta \), it can be seen that the chosen the performance index does not penalize for \( 2n\pi \) additions of \( \theta \), that is, different equilibria can be achieved with the same cost. Now \( q_i \) and \( r \) are the design variables. For the design, we chose

\[ q_1 = 10 \quad q_2 = 0.1 \quad q_3 = 0.1 \quad q_4 = 0.1 \quad r = 1 \]  

(22)

which yields state and control weightings of

\[ Q(x) = \begin{bmatrix}
  10 & 0 & 0 & 0 \\
  0 & 0.1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & \frac{1}{(1-x_3^2)\Delta}
\end{bmatrix} \]  

(23)

This selection of weighting penalties resulted in reasonable control usage (near maximum for maximum cart initial position) and provided good settling times for all initial conditions tested.

The SDRE technique requires that the first order equation be brought to state-dependent coefficient (SDC) form, that is,

\[ \dot{x} = f(x) + g(x)u \]  

\[ = A(x)x + B(x)u \]  

(24)

It has been shown [2] that there are an infinite number of ways to accomplish this. For guaranteed local asymptotic stability, we seek SDC parameterizations that are at least stabilizable and detectable in the region of interest. Out of the set of stabilizable and detectable parameterizations, some parameterizations may perform better than others in terms of suboptimal performance. One parameterization was used for all simulation results. The SDC parameterization chosen was

\[ A(x) = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  -\frac{1}{\Delta} & 0 & \frac{e_x^2}{(1-x_3^2)\Delta} & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{e(1-x_3^2)}{\Delta} & 0 & 0 & -\frac{x_4^2}{(1-x_3^2)\Delta}
\end{bmatrix} \]  

(25)

with \( B(x) = g(x) \).

4. Design Results

A comparison was made between the optimal open-loop control and the SDRE nonlinear feedback control for a wide range of initial conditions. The optimal control solution was found using a conjugate gradient algorithm. The conjugate gradient algorithm is a first order method using the function and its gradient. The
algorithm is based on a variational approach and generates the optimal control solution as a trajectory in time and not in feedback form. This method can only find the optimal solution over a finite time interval. This interval was chosen large enough (25 nondimensional time units) so that dynamics had plenty of time to reach an equilibrium position. Thus, the results numerically represent the optimal solution over an infinite horizon.

The SDRE control is obtained in the nonlinear feedback form

$$u = -r^{-1}B^T(z)P(x)x$$

(26)

where $P(x)$ is the positive (semi)definite solution of

$$A^T(z)P + PA(z) - PB^T(z)r^{-1}(x)B(x)P + Q(x) = 0$$

(27)

In the simulation the SDRE control was solved on-line using Matlab calls within Simulink. The Ricatti equation was solved using the Matlab function “are”. It should be noted that controllability is lost for $\cos \theta = 0$. This condition was handled by establishing a small region, $|\cos \theta| \leq 10^{-5}$ around $\cos \theta = 0$ and setting the control to zero inside this region. Since there is no friction, the angular velocity of the pendulum moves the pendulum through this region without the need of control.

The first result is for an initial cart position of .3 and zero initial velocity. Figure 2 shows the control solution. Notice how the SDRE solution is very similar to the optimal solution. Figure 3 shows the phase portrait of the bob position and velocity. Figure 4 shows the time histories of the cart position and velocity. The performance index for the SDRE algorithm is 2.1502; for the conjugate gradient it is 2.1271. Thus, the SDRE nonlinear feedback control solution is 1.1% from optimal. This is a typical result starting near the origin.

Two difficulties arise when the cart is far from the origin near the maximal allowable position. The first difficulty is the need for more control effort than the bound allows. This brings up the question of placing a hard limit on the control variable as will be done in the next design iteration or increasing the control penalty weight $r$ in the performance index (20). Increasing $r$ would avoid actuator saturation for these extreme initial positions but at the expense of slower response time. Instead, we simply left $r = 1$ and inserted a control magnitude limiter into the simulation. There are two justifications for doing so. First, for the large majority of feasible initial conditions, no control saturation occurs with $r = 1$. Second, in the absence of uncertainties and disturbances it can be proven that for a single control variable, the optimal bounded open loop control is equivalent to the optimal unbounded feedback control (which may be unknown) without saturation limits imposed. If the unbounded SDRE feedback solution is sufficiently close to the optimal unbounded open loop control, then the SDRE feedback solution with saturation limits imposed will be close to the optimal bounded open loop control. The conjugate gradient algorithm found the optimal bounded open loop control by introducing an algebraic constraint that included the control and a slack variable. The solutions obtained from the two methods, i.e., SDRE with imposed saturation limits and the conjugate gradient algorithm with algebraic constraint, are compared in Figures 6 through 8. In this case, the initial conditions are $x_1 = 1$, $x_2 = .5$. It should be noted that by using the SDRE unbounded feedback solution with a posteriori imposed saturated limits instead of directly incorporating a hard constraint on the control into the problem formulation as in [2] keeps the complexity of the design low.

The second difficulty is the advent of multiple local minima as the distance from the origin is increased. This difficulty did not pose a problem for the SDRE design. Indeed, using the chosen SDC parameterization (25), the SDRE control consistently went to the same solution. Additionally, it is conjectured that a different parameterization can lead to a different equilibrium. This has been shown for some initial conditions on this problem, but since a single parameterization is desired, these results are not presented here. Multiple local minima also exist as the initial velocity is increased. This characteristic of the problem is illustrated for an initial cart position and velocity of .75 and 1.0, respectively; using two different starting points, the conjugate gradient algorithm found two local optima. The first solution has a smaller overall cost than the first, but by less than 1%. The results are shown in Figures 9 through 11. One solution returns to the origin, the other to the equilibrium at $x_3 = \pi$. Note that the two locally optimal controls are approximately 180 degrees out of phase (Figure 9) yet basically produce the same cart motion (Figure 11). For clarification, Figure 11 contains six plots; the cart position and velocity responses from the two locally optimal controls and from the SDRE control. The maximum initial cart velocity that did not result in the SDRE algorithm going to the equilibrium at $x_3 = \pi$ was determined to be approximately .7 for this particular cost function.

The overall comparison of the SDRE method is presented in Table 1. The table lists the SDRE cost over optimal cost given the same initial conditions and same ending equilibrium condition. The initial conditions of the bob are $\theta = 0$ and $\dot{\theta} = 0$. As can be seen the SDRE algorithm provides excellent results for a wide variety of initial conditions with the largest difference from optimal being only 17.5%.
The SDRE closed-loop system for this problem acts as a linear system in steady state with respect to small amplitude input disturbances. This can be seen in Figure 12 which depicts the cart response to a 5 radians/sec sinusoidal disturbance. After the initial transient has subsided, the response contains only the input frequency. This steady-state linear behavior is independent of initial conditions and holds for disturbances of different frequencies as well as for disturbances containing multiple frequencies; the steady-state frequency content of the response is the same as that of the input. This behavior allows us to conduct steady-state frequency analysis of the disturbance rejection aspects of the closed-loop system. Here, in addition to the SDRE regulator solution (nonlinear $H_2$), we also investigated the effects of using the SDRE method within a full state information nonlinear $H_\infty$ formulation [2]. This simply entailed incorporating a $\gamma B_1(z)B_1(z)^T$ term into the state-dependent Riccati equation, where $B_1(z) = d(z)$ with $d(z)$ defined in (17)-(18). A constant value of $\gamma = 6.28$ was chosen to enhance worse-case disturbance effects. Figures 13 and 14 show the normalized steady-state frequency response of the cart position and cart velocity; a magnitude of less than one indicates attenuation. From Figure 13 it can be seen that the SDRE nonlinear $H_2$ design attenuates middle-frequency disturbances in the range of 2 - 10 radians/sec. It can also be seen that the SDRE nonlinear $H_\infty$ controller attempts to flatten out the frequency response of the cart position at the expense of increased control activity as evidenced in Figures 15 and 16. It should be noted that even though disturbance attenuation is not achieved for frequencies less than 2 radians/sec, the closed-loop system is still stable throughout that region whereas disturbances around 1 radian/sec destabilize the open-loop system.

Finally, Figure 17 shows how the SDRE controller is able to handle parameter variations. The plant is varied by changing the coupling parameter plus and minus 25%, that is $\epsilon = .25$ in one case and $\epsilon = .15$ in the other. The variation of the coupling parameter is not an actual plant variation since this parameter cannot change without changing physical characteristics that have other effects on the nondimensional plant model. Nevertheless, the controller does an excellent job of handling the variation.

<table>
<thead>
<tr>
<th>Velocity</th>
<th>Position</th>
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<tbody>
<tr>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.5000</td>
<td>6.4988</td>
</tr>
<tr>
<td>1.0000</td>
<td>33.6717</td>
</tr>
</tbody>
</table>

Table 1. Performance Cost of SDRE Control Versus Optimal Control
Figure 4: Cart Position and Velocity

Figure 7: Cart Position and Velocity

Figure 5: Control Solution

Figure 8: Control Solution for Two Local Optima

Figure 6: Pendulum Bob Phase Plane Response

Figure 9: Pendulum Bob Phase Plane Response
Figure 10: Cart Position and Velocity

Figure 11: Steady-State Linear Behavior

Figure 12: Disturbance Attenuation

Figure 13: Disturbance Attenuation

Figure 14: Actuator Activity

Figure 15: Actuator Activity
5. Summary

A control design of the nonlinear benchmark problem has been carried out using the state-dependent Riccati equation method. A state space representation was employed that allowed the controller not to distinguish between $2\pi n$ additions to $\theta$, the angular position of the pendulum. It was demonstrated that the SDRE nonlinear feedback solution compares favorably to the open-loop optimal solution in the absence of disturbances and uncertainties and is robust to parametric variations and disturbances.

References


