Abstract

In this paper we used the Linear Fractional Representation (LFR) toolbox for the numeric generation of an LFT model. The LFT model was used for structural aircraft analysis in order to determine the onset of flutter. The equation of motion for structural aircraft response is first parametrised in terms of Mach number to obtain matched flutter solutions. From this parametrisation, LFT models for flutter analysis were analytically and numerically generated. The flutter analysis results of the LFT models is compared with flight test results. Finally, an LFT model is generated with perturbations in Mach number, mass, damping and stiffness for robust flutter analysis. The results from this analysis are compared with flight test data and correlate better than unperturbed nominal model analysis.

1 Introduction

A critical structural analysis topic in the design of aircraft is flutter analysis of flutter. Consider the dynamic motion of the complete aircraft structure with emphasis on the lift generating surfaces, such as wings and tail surfaces together with the different external store configurations. Flutter analysis relies on the solution of the equation of motion for structural aircraft response (1), which at a constant Mach number Ma is given as:

\[ M_\eta s^2 + C_\eta + K_\eta = qQ_\eta \quad (1) \]

with \( M, C \) and \( K \) to denote the generalized mass, damping and stiffness matrices. These result from the structural mode shape analysis using a Finite Element Model (FEM) of the system under consideration. The FEM model is analysed using the software package NASTRAN [7, 3]. The term \( q = \frac{1}{2} \rho V^2 \) represents the dynamic pressure at which the system is considered. To obtain critical flutter speeds the nominal dynamics of the aeroelastic system given by equation (1) are generally solved in the frequency domain using methods known as the \( \mathcal{P} \), the \( \mathcal{K} \) and the \( \mathcal{P-K} \) method, see [1].

The unsteady aerodynamic forces acting on the right hand side of the system (1) are determined in terms of a frequency dependent matrix \( Q(k, Ma) \), where \( k \) is the reduced frequency. These are obtained using the Doublet-Lattice Method (DLM) which is incorporated in the software package NASTRAN. The DLM is a linear aerodynamic theory applicable up to subsonic airflow. The Mach number for which this method produces valid results is limited to \( Ma = 0.8 \). With the speed of sound \( V_s \) the Mach number is defined as \( Ma = \frac{V}{V_s} \). The reduced frequency \( k \) is defined as \( k = \frac{\omega}{V_s} \). For anti-symmetric modes it depends on the radial frequency \( \omega \), the wing span \( b \) and the true air speed \( V \). The frequency \( \omega \) belongs to a certain mode shape, and the coefficients in the matrix \( Q \) reflect the aerodynamic influence between the different mode shapes at the matching reduced frequency \( k_i \).

Assuming that (1) has pure harmonic solutions we introduce the complex Laplace variable \( s = \omega \) such that the unsteady aerodynamics \( Q(k, Ma) \) are transformed into frequency dependent dynamics \( G(s, Ma) \) by means of function approximation. The unsteady aerodynamic forces can be modeled with \( \mathcal{I} \)-lag terms such that \( Q(s, Ma) \) is reformulated as:

\[
\begin{bmatrix}
A_0 + A_1 s & b & A_2 s^2 \left( \frac{b}{2V} \right)^2 + \sum_{i=1}^{L} A_{2+i} s + \beta_i \frac{s}{b} \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} (2)
\]

where the coefficients \( A_0, A_1 \), and \( A_2 \) reflect quasi-steady aerodynamic contributions to denote respectively the aerodynamic stiffness, damping and inertia. The unsteady aerodynamic contributions in the matrix \( Q \) are actually modeled by the means of pure aerodynamic Padé delays. These delays are scaled with \( A_{2+i} \) and have a pole at \(-\beta_i \frac{2V}{b}\) for each lag term in equation (2). Defining each of the aerodynamic lag modes as a new state \( x = \frac{1}{\beta_i S + \frac{2V}{b}} \) leads to a new set of equations:

\[
\begin{align*}
\ddot{\eta} + \dot{\eta} + K \eta &= q \sum_{i=1}^{L} A_{2+i} [x] \\
\dot{z}_i &= \dot{\eta} - \beta_i \frac{2V}{b} x_i \\
\end{align*}
\]

with \( \bar{M}, \bar{C} \) and \( \bar{K} \) defined as:

\[
\bar{M} = M - A_2 \left( \frac{b}{2V} \right)^2 q
\]

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\[ \mathcal{C} = C - A_1 \left( \frac{b}{2V} \right) q \]  
\[ \mathcal{K} = K - A_0 q \]  
\[ \text{(6)} \]
\[ \text{(7)} \]

In order to perform parametric robustness analysis in the \( \mu \)-framework we seek for a state-space representation of the equation of motion (3). From the state-space representation one can generate a parametric Linear Fractional Transformation (LFT) of the original system affected by variations in dynamic pressure \( q \) in order to determine the nominal flutter margin. For robust flutter margin detection the apparent mass \( \mathcal{M} \), damping \( \mathcal{K} \) and stiffness \( \mathcal{C} \) are further allowed to vary simultaneously.

Equations (4) and (3) are first reformulated into state space form, with generalised states \( \eta, \eta, \eta \) and aerodynamic states \( z \). The set of first order differential equations is given as:

\[
\begin{bmatrix}
\eta \\
\eta \\
\eta \\
z_1 \\
z_t
\end{bmatrix} = -A
\begin{bmatrix}
\eta \\
\eta \\
\eta \\
z_1 \\
z_t
\end{bmatrix}
\]  
\[ \text{(8)} \]

with \( A \) defined as:

\[
\begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
-M^{-1}K & -M^{-1}C & M^{-1}A_3q & \cdots & M^{-1}A_{2+q} \\
0 & I & -B_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & I & 0 & \cdots & -B_t
\end{bmatrix}
\]  
\[ \text{(9)} \]

in which \( B_t \) is defined as:

\[
B_t = \left( \frac{b}{2V} \right)
\begin{bmatrix}
\beta_t & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \beta_t
\end{bmatrix}
\]  
\[ \text{(10)} \]

The flutter point is determined when (8) becomes singular as the dynamic pressure is varied towards the flutter pressure \( q_f \) to which a flutter speed \( V_f \) is associated. This flutter speed occurs at another Mach number \( M_{a_f} \) than the Mach number \( M_a \) at which the model was derived in 1. One speaks of unmatched flutter speed. The matched flutter speed or pressure is obtained iteratively over \( M_{a_u} \) until the condition \( M_{a_u} = M_{a_f} \) is achieved.

\[ \text{2 Matched flutter Condition} \]

All the flutter solutions to the K-, PK- and P method, lead to unmatched flutter solutions. Especially, for flight test purposes it is desirable to obtain directly, without any iteration process matched flight flutter solutions. A suitable flight test procedure is the level acceleration. The flight is performed at constant flight levels starting from a minimum speed, for instance at \( 1.2 V_{\text{min}} \). While maintaining level flight, the velocity is increased by means of engine power control. The flight test run ends at a predetermined speed, for example the maximum airspeed \( V_{\text{max}} \) to which corresponds the maximum Mach number \( M_{a_{\text{max}}} \).

These flight tests can only be performed in a safe manner, when the test crew is provided with adequate flutter speed values. From the way these tests are conducted it is desirable to know on-line the matched flutter solutions in order not to exceed the flutter Mach number \( M_{a_f} \) at a fixed flight level. At constant flight level the air density \( \rho \) and the speed of sound \( V_s \) are fixed in the state-space model (8). This means that for level flight all the system variations can now be expressed as a function of the Mach number only. The flutter solution when the system is reparametrised in terms of the Mach number, at a considered flight level is automatically a matched flutter solution.

We now reformulate the state space system matrix \( A \) in (8) as a function of Mach number only. The matrix coefficients \( A_i \) from the unsteady aerodynamic influence matrix \( Q \) are determined as a function of \( k \) at a selected Mach number, are first expressed as a function of Mach. These matrix coefficients are approximated using a second order polynomial fit in the Mach number:

\[
A_i = A_{i_0} M_{a_f}^2 + A_{i_1} M_{a_f} + A_{i_0}
\]  
\[ \text{(11)} \]

Terms depending on airspeed such as \( \frac{b}{2V} \) and the dynamic pressure \( q \) are reformulated in function of the Mach number. The airspeed \( V \) in terms of Mach and speed of sound is \( V = M V_s \), so that we obtain for the expression \( \frac{b}{2V} = \frac{b}{2M V_s} \). For the dynamic pressure \( q = \frac{1}{2} \rho V^2 = \frac{1}{2} \rho (M V_s)^2 \). Expressing the matrices \( \mathcal{M}, \mathcal{C} \) and \( \mathcal{K} \) in terms of the Mach number leads to:

\[
\mathcal{M} = M - A_2 \left( \frac{b}{2V} \right)^2 q
\]  
\[ \text{(12)} \]
\[
= M - \frac{b^2 \rho}{8} A_2 (M a)
\]

with

\[
A_2 (M a) = A_{2_0} M_{a_f}^2 + A_{2_1} M_{a_f} + A_{2_0}
\]  
\[ \text{(13)} \]
\[
\mathcal{C} = C - A_1 \left( \frac{b}{2V} \right) q
\]  
\[ \text{(14)} \]
\[ C = \frac{-bpMaV_s}{4} A_1(Ma) \]
with
\[ A_1(Ma) = A_1 m_2 Ma^2 + A_1 m_1 Ma + A_1 m_0 (15) \]
\[ \bar{K} = K - A_0 q \]
\[ = K - \frac{\rho Ma^2 V_s^2}{2} A_0(Ma) \]
with
\[ A_0(Ma) = A_0 m_2 Ma^2 + A_0 m_1 Ma + A_0 m_0 (17) \]
All coefficients of equation (8) are now expressed as a function of Mach and correspond to a certain altitude fixing the air density \( \rho \) and the speed of sound \( V_s \) to constant values.

The advantage of the described method is that the matched flight flutter condition is obtained solving an eigenvalue problem for (8) without any iteration process. The difficulty in the method lies in the extensive curve fitting of the coefficients of \( Q(k, Ma) \) in order to obtain polynomial expressions in the Mach number. The resulting Mach dependent state-space equations can be automatically reformulated in an LFT format for the purpose of robustness analysis.

An aero-elastic stability analysis of using the P Mach matched method was first performed and the results are shown in figure 1. Also a Mach perturbed LFT model was constructed with the LFR toolbox and the stability of the numerically obtained system was analysed for varying Mach numbers.

We have modeled the LFT for the Mach number variation as an additive uncertainty with nominal value \( Ma_0 \) and uncertainty \( \delta Ma \) such that \( Ma = Ma_0 + \delta Ma \). When equations (13), (15) and (17) are implemented in equation (9) elements arise with Mach dependency up to powers of \( Ma^4 \). In this way, many high order Mach dependent cross products terms emerge for which it is hard to derive analytically the corresponding LFT model [4]. The desired LFT model was numerically obtained using the LFR toolbox [5].

The flutter margin is expressed as a matched Mach number margin valid at the considered flight level. The flutter Mach numbers from both methods are summarised table 1. The results of the P Mach matched method, presented in fig 1, show the frequency and damping curves versus the Mach number at sea level.

The graph shows the aero-elastic instability of the first wing bending and torsion mode. The curves show good correlation with earlier obtained results of the PK and P method on a model parametrized in terms of dynamic pressure. The damping versus Mach number graph shows the instability of the torsion mode at a Mach number of 0.7. Table 1 shows that the results of the P Mach matched method and the Mach perturbed LFT correlate well. The difference in the flutter numbers is due to numerical round off errors. The good correlation between the methods proves the reliability of the P Mach matched method and strengthens the confidence in the numerical LFT model generated using the LFR toolbox.

![Figure 1: Frequency and Damping Results of P Mach matched method at Sea Level.](image)

### Table 1: Comparison of P-method and Numerically LFT model Results at Sea Level

<table>
<thead>
<tr>
<th>P Mach matched method</th>
<th>Mach perturbed LFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ma_{flutter} )</td>
<td>( Ma_{flutter} )</td>
</tr>
<tr>
<td>0.705</td>
<td>0.698</td>
</tr>
</tbody>
</table>

The robust LFT models are developed to reflect simultaneous uncertainties with respect to mass, damping and stiffness. All LFT models have the Mach number as varying parameter for flutter analysis. The obtained flight flutter margin is a robust Mach matched solution valid at a selected altitude. The LFT generation requires about half an hour on a Pentium 700 MHz.

### Mass Perturbation

Equation (8) indicates that the mass parameter enters the state space equation in an inverse manner. The LFR toolbox requires the parameter that is to be perturbed, defined linear or in higher order power terms,
but not as inverse terms. The mass perturbation $M$ was therefore defined as $\Delta M = M_0 + M$. The fuel contents of the wing pylon fuel tanks was chosen for the mass perturbation, because a large influence of pylon fuel contents was indicated on Limit Cycle Oscillation (LCO) onset in an NLR analysis [6] and confirmed during flight tests. The fuel tank content was altered from full (2600 lbs per tank) down to empty in two intermediate steps, between, i.e. half full and quarter full. The wing pylon fuel tank contains three fuel compartments, which can be drained in sequence.

A low cost design change, the resequencing of the wing pylon fuel tank, was initiated by General Dynamics to alleviate the LCO problem after flight flutter testing [2]. The sequence is from Full (F), indicated as FFF, only fuel is used from the mid compartment. In the half full state, indicated as FEF, only the forward and aft compartments contain fuel. After that, fuel is used from the aft compartment. In the quarter full state, indicated as FEE, only the forward compartment contains fuel. When the forward compartment is drained, the tank is Empty (E), which is indicated as EEE. Obviously such a fuel sequence causes an irregular change of centre of gravity of the fuel tank resulting in a non-linear change of the moment of inertia of the wing with respect to change in fuel state of the wing pylon fuel tank. The modelling of the fuel state in the wing pylon fuel tank was divided into four sequences, denoted with the FFF, FEF, FEE and EEE states.

Flutter analysis of all the perturbed fuel states of the wing pylon fuel tank was performed using the mass perturbed LFT model and the P Mach matched method. The robust analysis was performed with a nominal Mach number of 0.6.

The perturbation increment in the Mach number is 0.1, and the model was fitted to a function between Mach 0.4 and 0.9.

The results of the analysis are presented in figure 2 reflecting the total mass perturbation in the wing pylon fuel tank from full FFF down to empty EEE.

Analysis shows an increase of the flutter speed from fuel state FFF and FEE with decreasing fuel in the wing pylon fuel tank. Again, the solutions deviate at larger perturbations in the fuel state, however the largest deviation was in the third perturbation run and did not exceed 5 percent. The third run was conducted to 500 lbs perturbation from the FEE fuel state. The last two perturbations of 600 and 700 lbs showed a flutter speed above a Mach number of 0.9 for both methods. These solutions were considered not to be valid, because both models were fitted as a function of Mach from 0.4 up to 0.9.

Frequency and damping curves of the P Mach matched method have been generated as a function of Mach number for the FFF to EEE fuel state. In the FFF fuel state, the aero-elastic instability of the wing first bending and torsion mode, shows classical flutter behaviour as the torsion mode becomes unstable at a Mach number of 0.7. In the FEF fuel state the torsion mode becomes unstable at a Mach number of 0.58, but the damping curve crosses the zero damping axis at a very shallow angle. The damping of the torsion mode decreases very slowly indicating that the aero-elastic problem is not violent. The frequency and damping curves of the FEE fuel state show a similar behaviour as in the FEF fuel state. With empty wing pylon fuel tanks EEE the damping of the bending and torsion modes remain positive up to a Mach number of 0.9 which is beyond the validity region of the model.

The perturbation of the fuel contents of the wing fuel pylon had to be carried out in three runs, due to the irregular fuel drainage scheme of the tanks. The solution of the LFT model showed good correlation with the P Mach matched method, with a maximum deviation between the robust model and the P method of 7 percent. To limit the difference between the solutions, the actual mass perturbation has to be limited. How much this limitation should be is difficult to say, because the difference is non-linearly distributed with increasing mass perturbation. However, based on the total mass perturbation in figure 2 and a maximum difference of 5 percent between the solutions, the maximum perturbation should be 1000 lbs from the FFF fuel state. For the FEF and FEE fuel states there are no limits for the mass perturbation, because the difference between the methods never exceeded 5 percent. These theoretical mass perturbation limits are not considered a problem, because they are way beyond the practical mass perturbation limit. The practical limit is related to the

![Figure 2: Flutter Analysis with Mass Perturbation](image-url)
accuracy of weighing the aircraft and the error with which the fuel state in the wing pylon fuel tank can be determined. To summarise both influences, a practical perturbation limit on the wing pylon fuel state mass perturbation of +/- 200 lbs was established. The influence of this practical perturbation limit of +/- 200 lbs is the highest in the third perturbation run, from FEE to EEE, where it creates a variation of the flutter Mach number of 31 percent.

4 Validation of the Robust LFT Model using Flight Test Results

Flight Test data from the modelled configuration was made available by the RNLAF. The flight test data consist of a collection of relevant parameters, recorded during a level acceleration at a selected altitude. The flutter analysis was performed at sea level, where the dynamic pressure is the highest possible. This creates than a worse case flutter solution. Level acceleration at sea level in practise is not possible due to restrictions, however, a level acceleration at 1000 ft AGL is a small deviation from sea level. The data that is available is from runs above the North sea with different fuel states.

Figures 3 and 4 show time histories of flight test results from level accelerations. From top to bottom we have the normal acceleration of the forward and the aft station of left wing tip a_{lf}, a_{la} in m/s^2, and the normal acceleration of the forward and aft station of right wing tip a_{rf}, a_{ra} and finally the Mach number Ma as function of time.

The runs are started from 250 KCAS up to the limit V_{Ma}. During the run, the altitude is closely maintained with a margin of 50 ft. Also, the fuel state of the wing pylon fuel tank was frozen by selecting internal fuel feed. With this selection, fuel is used from the internal tanks instead of fuel from the external tanks. This was done to avoid a variable fuel state in the wing pylon fuel tanks, which might influence the measurement. The graphs show the emerging vibration of the wingtips, which stabilise at about 20m/s^2 for the forward wingtip stations. It is obvious why the vibrations are called LCO instead of flutter since these do not diverge. The LCO from flight starts to show at 16 seconds in the run and vibrations of the wingtips emerge at a Mach number of 0.61. Similarly from figure 4 an LCO onset speed of Mach 0.57 is indicated. The acceleration data of the wingtips indicate that the forward and aft stations are 180° out of phase (left forward station goes up, while left aft station goes down), i.e. the motion of the wingtips is mainly torsional. Data analysis also shows that both forward stations are 180° out of phase (left wingtip goes up, while the right wingtip goes down), i.e. the motion is anti-symmetric. Fourier analysis of all acceleration signals reveal a peak in the power spectral density at 5 Hz, indicating that the aero-elastic instability is an anti-symmetric wing torsion mode with a frequency of 5 Hz.

5 Conclusion

The LCO onset Mach numbers are compared with the flutter Mach numbers from the robust models. To establish the lowest and the highest possible flutter Mach number for the associated configuration we assume that the fuel state of the wing pylon fuel tanks can be determined with an accuracy of +/- 200 lbs. The first run was executed with 2000 lbs per wing pylon fuel tank. This is a perturbation of 600 lbs from the full fuel state FFF. With the uncertainty of +/- 200 lbs, we have a total range of 400 to 800 lbs perturbation from the full fuel state FFF. Similarly for the second run, which was executed with 1000 lbs per wing pylon fuel tank, we establish a total range of 1400 to 1800 lbs perturbation from the full fuel state FFF. Studying damping variation show that the lowest flutter Mach number is at zero structural damping variation, while the highest flutter Mach number is at 0.03 structural damping variation. Stiffness perturbation analysis reveals that the lowest flutter Mach number is at -10 percent stiffness perturbation, while the highest flutter Mach number is at +10 percent stiffness perturbation. With these perturbations, a lowest and a highest possible flutter Mach number was calculated using the robust LFT model. The results of this robustness analysis are summarised in table 2.

<table>
<thead>
<tr>
<th>Fuel State [lbs/tank]</th>
<th>Rob. Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>1000</td>
</tr>
<tr>
<td>400</td>
<td>800</td>
</tr>
<tr>
<td>1400</td>
<td>1800</td>
</tr>
<tr>
<td>0.03</td>
<td>0</td>
</tr>
<tr>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>0.79</td>
<td>0.6</td>
</tr>
<tr>
<td>0.57</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Table 2: Results of Robustness Analysis and Flight Tests

The results from the robustness analysis in table 2 indicate for the first run, with a fuel state of 2000 lbs per wing pylon fuel tank, a robust flutter Mach number between 0.6 and 0.79. From flight test data, the LCO onset Mach number was 0.61, which matches the robust flutter margin. Similarly, the second run with a fuel state of 1000 lbs per wing pylon fuel tank, the robustness analysis results indicate a flutter Mach number between 0.57 and 0.85. From flight test data, the LCO onset Mach number was 0.57, which also matches the robust flutter margin. The comparison between robustness analysis and flight test data results indicate that the nominal models are too optimistic as far as the
flutter Mach number concerns. When perturbations in mass, damping and stiffness are taken into account, a more realistic margin of a flutter Mach number is produced. The robustness analysis may also explain the deviation in LCO onset Mach number between different airframes. Aero-elastic analysis indicated an instability of the anti-symmetric wing torsion mode of 5 Hz. This mode was also confirmed by frequency and phase analysis of the flight test data.

Acknowledgment

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References


