Computer-Aided Uncertainty Modeling of Nonlinear Parameter-Dependent Systems, Part II: F-16 Example

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Abstract

Robust control system analysis and design is based on an uncertainty description, called a linear fractional transformation (LFT), which separates the uncertain (or varying) part of the system from the nominal system. System uncertainties to be represented in LFT form include both parametric and non-parametric uncertainties for many practical problems. LFT formulation for these "mixed uncertainty" systems involves: formulation of a linear parameter varying (LPV) model, construction of a low-order parametric LFT model, and validation of a mixed uncertainty model with respect to measurement data. This paper presents an F-16 aircraft example problem that illustrates the development of a validated mixed uncertainty model for this vehicle.

1. Introduction

Robust control problems require the formulation of a model that characterizes the nominal and uncertain parts of the system, called a P-Δ model, as depicted in Figure 1.

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} \begin{bmatrix}
x \\
u
\end{bmatrix} + \begin{bmatrix}
0 \\
I
\end{bmatrix} w_{\Delta}
\]

Figure 1. Robust Control System Block Diagram

Block matrix \( P \) (partitioned as \( P_{11}, P_{12}, P_{21}, \) and \( P_{22} \)) represents the generalized nominal plant, \( \Delta \) is an uncertainty matrix, \( x \) is the state vector, \( u \) is the control input vector, and \( y \) is a vector of measurement signals. Signals \( w_{\Delta} \) and \( z_{\Delta} \) provide the connections between the uncertainties, \( \Delta \), and the plant, \( P \).

Systems whose models contain both parametric and non-parametric uncertainties will be referred to in this paper as "mixed uncertainty" systems. The uncertainty block, \( \Delta \), for mixed uncertainty systems represents both parametric and non-parametric uncertainties, and can therefore contain real scalar parameters and complex blocks. The uncertainty structure addressed in this paper is defined as follows.

\[
\Delta = \text{diag} [\Delta(\delta), \Delta_x]
\]

where: \( \Delta(\delta) = \text{diag} [\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_m I_{n_m}] \) (1.2a)

\[
\dim[\Delta(\delta)] = n_\Delta = \sum_{i=1}^{m} n_i
\] (1.2b)

\[
\delta = [\delta_1, \delta_2, \ldots, \delta_m] \in \mathbb{R}^m
\] (1.3)

\[
\Delta_x = \text{diag} [\Delta_1(s), \Delta_2(s), \ldots, \Delta_m(s)]
\] (1.4)

The process for formulating and validating P-Δ models for mixed uncertainty problems can be quite complicated, especially for systems with nonlinear parameter dependencies. An overview of three uncertainty modeling methods is given in Ref. [1] that can be used in combination to obtain a validated mixed uncertainty model for nonlinear systems involving both parametric and non-parametric uncertainties. The methods include: a multivariate polynomial modeling method based on orthogonal function modeling that can be used in developing a linear parameter varying (LPV) model of the system; a parametric uncertainty modeling method that computes a low-order linear fractional transformation (LFT) model based on the LPV model of the system; and an uncertainty bound identification method that can be used to obtain a validated mixed uncertainty model based on the parametric LFT model and assuming an uncertainty structure to characterize unmodeled system dynamics. These methods can be used in combination or separately for application to robust and LPV control system analysis and design problems.

This paper presents an example uncertainty modeling problem to illustrate the application of these methods. Section 2 summarizes development of the LPV model, Section 3 presents the computation of an equivalent low-order parametric LFT model, and Section 4 presents the development of a validated mixed uncertainty model for the system. Some concluding remarks are given in Section 5.

2. LPV Model Formulation

A nonlinear F-16 simulation containing wind tunnel data was used to generate a set of linear models at trim points throughout the vehicle flight envelope for various angles of attack (\( \alpha \)) and longitudinal center of gravity positions (\( x_{cg} \)), which were obtained as a fraction (percent) of the aircraft mean aerodynamic chord (MAC). The linear models generated over the parameter space consisted of the F-16 short period mode approximation, defined below.

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\phi}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
\alpha \\
\phi
\end{bmatrix} + \begin{bmatrix}
b_{11} \\
b_{21}
\end{bmatrix} u_{\text{elev}}
\] (2.1)
The parameter values over which the set of linear models of the form of Eqn. (2.1) were generated are defined as follows.

\[ \alpha = [5, 10, 15, 20, 25, 30, 35] \text{ deg} \quad (2.2a) \]
\[ x_{cg} = [0.20, 0.25, 0.30, 0.35] \%\text{MAC} \quad (2.2b) \]

The following nominal model was selected from the parameter space.

\[
\begin{bmatrix}
\alpha \\
\dot{q}
\end{bmatrix} = \begin{bmatrix}
-0.32858 & 0.93390 \\
-1.15408 & -0.66151
\end{bmatrix} + \begin{bmatrix}
\alpha \\
q
\end{bmatrix} + \begin{bmatrix}
-0.00071 \\
-0.03083
\end{bmatrix} u_{\text{elev}}
\]

where: \( \alpha_{\text{nom}} = 20 \text{ deg} \), \( x_{cg,\text{nom}} = 0.25 \)

Note that in the linear models, angle of attack (\( \alpha \)) and pitch rate (\( q \)) are in radians, while \( u_{\text{elev}} \) is in degrees.

In order to obtain an LPV model to characterize the vehicle over the parameter space of Eqns. (2.2), each matrix element in the set of linear models was parameterized using the orthogonal function modeling approach described in Ref. [1]. The LPV model form is given below,

\[
\begin{bmatrix}
\alpha \\
\dot{q}
\end{bmatrix} = \begin{bmatrix}
a_{11}(\delta) & a_{12}(\delta) \\
a_{21}(\delta) & a_{22}(\delta)
\end{bmatrix} + \begin{bmatrix}
b_{11}(\delta) \\
b_{21}(\delta)
\end{bmatrix} u_{\text{elev}} \quad (2.4)
\]

\[
\delta = [\delta_{\alpha}, \delta_{x_{cg}}] \quad (2.5)
\]

where \( \delta \) represents the uncertain parameter vector defined in Eqn. (1.3) prior to normalization. The parameterization was performed on the difference of each model in the set from the nominal model given in Eqn. (2.3). The parameterization results for each matrix element are given as follows.

\[
a_{11}(\delta) = a_{11,\text{nom}} + 0.80886 \delta_{\alpha} - 0.98722 \delta_{\alpha}^2 + 3.39948 \delta_{\alpha}^3 + 0.15370 \delta_{x_{cg}} \quad (2.6a)
\]
\[
a_{12}(\delta) = a_{12,\text{nom}} + 0.04742 \delta_{\alpha} - 0.30209 \delta_{\alpha}^3 \quad (2.6b)
\]
\[
a_{21}(\delta) = a_{21,\text{nom}} + 4.39386 \delta_{\alpha} + 21.97068 \delta_{\alpha}^2 - 561.40796 \delta_{\alpha}^3 + 9.60579 \delta_{x_{cg}}^2 + 55.68718 \delta_{x_{cg}} + 177.51148 \delta_{x_{cg}}^2 \quad (2.6c)
\]
\[
a_{22}(\delta) = a_{22,\text{nom}} + 1.20715 \delta_{\alpha} - 2.27408 \delta_{\alpha}^2 + 2.66061 \delta_{x_{cg}} - 5.10462 \delta_{x_{cg}}^2 \quad (2.6d)
\]
\[
b_{11}(\delta) = b_{11,\text{nom}} + 0.00175 \delta_{\alpha} - 0.00308 \delta_{\alpha}^2 + 0.00111 \delta_{x_{cg}} \quad (2.7a)
\]
\[
b_{21}(\delta) = b_{21,\text{nom}} + 0.07801 \delta_{\alpha} - 0.46425 \delta_{\alpha}^2 + 1.37262 \delta_{\alpha}^3 + 0.07119 \delta_{x_{cg}} \quad (2.7b)
\]

The resulting LPV model can be represented in compact state space form by the following equation.

\[
\dot{x} = \begin{bmatrix}
A(\delta) & B(\delta)
\end{bmatrix} \begin{bmatrix}
x \\
u
\end{bmatrix} = S(\delta) \begin{bmatrix}
x \\
u
\end{bmatrix} \quad (2.8)
\]

In standard P-\( \Delta \) model form, the \( \delta \) parameters are normalized. Scaling associated with the parameterization process and for normalizing the uncertain parameters can be accomplished as a separate step once the LFT model structure has been obtained. This is discussed further in Section 3.2.3.

3. Parametric LFT Model Formulation

The state-space parametric LFT model equations to be determined are given below

\[
z_{\Delta} = P_{11}w_{\Delta} + P_{22} \begin{bmatrix}
x \\
u
\end{bmatrix} \quad (3.1a)
\]
\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = P_{21}w_{\Delta} + P_{22} \begin{bmatrix}
x \\
u
\end{bmatrix} , w_{\Delta} = \Delta z_{\Delta} \quad (3.2b)
\]

where \( \Delta \) is the uncertainty matrix. Determining a solution for \( P_{11}, P_{12}, P_{21}, \) and \( P_{22} \) such that the resulting uncertainty model is low-order is the objective of this section.

3.1 Parametric LFT Problem Formulation

In order to obtain an LFT model of the LPV system, an LFT problem must first be formed. The following LFT equation is defined relative to the non-scaled parameters, \( \delta \).

\[
S(\delta) = P_{21}(I - \Delta P_{11})^{-1} \Delta P_{12} + P_{22} \quad (3.3)
\]
\[
\Rightarrow \quad S(\delta) = S(\delta) + S_0 \quad (3.4)
\]

For this example, it is simple to separate the nominal component (\( P_{22} \)) and the uncertain component that depends on \( \delta \). The results are given below

\[
S_0 = \begin{bmatrix}
A_0 & B_0
\end{bmatrix} = P_{22} \quad (3.5)
\]

\[
S_{\Delta}(\delta) = P_{21}(I - P_{11} \Delta)^{-1} \Delta P_{12} \quad (3.6)
\]

where:

\[
S_{\Delta}(\delta) = \begin{bmatrix}
a_{11}(\delta) & a_{12}(\delta) & b_{11}(\delta) \\
21(\delta) & a_{22}(\delta) & b_{21}(\delta)
\end{bmatrix} \quad (3.7)
\]

and the elements of \( S_{\Delta}(\delta) \) in Eqn. (3.7) are given in Eqns. (2.6) – (2.7). The \( A_0 \) and \( B_0 \) matrices of Eqn. (3.5) are defined in Eqn. (2.3a), i.e.:

\[
A_0 = \begin{bmatrix}
-0.32858 & 0.93390 \\
-1.15408 & -0.66151
\end{bmatrix} , B_0 = \begin{bmatrix}
-0.00071 \\
-0.03083
\end{bmatrix} \quad (3.8)
\]
3.2 Parametric LFT Model Construction

Once Eqns. (3.5) and (3.7) have been formed for the uncertain system, the LFT model can be obtained by solving Eqn. (3.6) for $B_1, p_{1z},$ and $\ell_{1z}$ such that the dimension of the $\Delta$ matrix is as small as possible. An overview of a numerical LFT modeling method for solving Eqn. (3.6) for multivariate polynomial problems is given in Ref. [1], as well as a technique for applying this numerical method to solving multivariate rational problems.

$S_A(\delta)$ in Eqn. (3.7) can be expanded as the sum of multivariate polynomial terms and the associated matrix coefficients, as shown below.

$$S_A(\delta) = S_{308a} \delta_{30} + S_{308c,d} \delta_{d} + S_{318a} \delta_{31}^2 + S_{318c,d} \delta_{d}^2 + S_{328a} \delta_{32} + S_{328c,d} \delta_{d}^3 + S_{338a} \delta_{33} \delta_{d} + S_{338c,d} \delta_{d}^4 + S_{348a} \delta_{34} \delta_{d}^3 + S_{348c,d} \delta_{d}^4$$ (3.10)

where:

$$S_{308a} = \begin{bmatrix} 0.80886 & 0.04742 & 0.00175 \\ 0.15370 & 0 & 0.00308 \\ 9.60579 & 2.66061 & 0.07719 \end{bmatrix}$$ (3.11)

$$S_{308c,d} = \begin{bmatrix} -0.98722 & 0 & -0.00308 \\ 21.97068 & -2.27408 & -0.46425 \end{bmatrix}$$ (3.12)

$$S_{318a} = \begin{bmatrix} 3.39948 & -0.30209 & 0 \\ 0 & 0 & 1.37262 \end{bmatrix}$$ (3.13)

$$S_{328a} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (3.14)

$$S_{338a} = \begin{bmatrix} 3.39948 & -0.30209 & 0 \\ 0 & 0 & 1.37262 \end{bmatrix}$$ (3.15)

$$S_{348a} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$ (3.16)

Equation (3.9) can also be written in terms of the $P_{11},$ $P_{12},$ and $P_{21}$ matrices as follows.

$$S_A(\delta) = P_{21} \Delta(\delta) P_{12} + P_{21}(\Delta P_{11} + (\Delta P_{11})^2 + ... + (\Delta P_{11})^r) \Delta(\delta) P_{12}$$ (3.19)

The first term on the right side of Eqn. (3.19) represents the linear uncertain components of $S_A(\delta),$ and the remaining terms are nonlinear. The order ($r$) of the highest nonzero term in the series of Eqn. (3.19) is determined from the degree of the highest term appearing in $S_A(\delta).$ This is determined from Eqns. (3.9) - (3.18) as follows.

$$r = \text{degree} (\delta_A^4) - 1 = (4) - 1 = 3$$ (3.21)

Thus, the exponent $r$ in Eqns. (3.19) - (3.20) equals 3.

The $P_{11},$ $P_{12},$ and $P_{21}$ matrices can be partitioned for each uncertain parameter as follows.

$$P_{11} = \begin{bmatrix} P_{118} & \delta_A \alpha \\ P_{118} & \delta_A x_{cg} \end{bmatrix}$$ (3.22)

$$P_{12} = \begin{bmatrix} P_{128} & \alpha \\ P_{128} & x_{cg} \end{bmatrix}$$ (3.23)

$$P_{21} = \begin{bmatrix} P_{218} \alpha \\ P_{218} x_{cg} \end{bmatrix}$$ (3.24)

where:

$$P_{118} \in R^{n_{rows} \times n_{cols}}, P_{128} \in R^{n_{rows} \times n_{i}}, P_{218} \in R^{n_{rows} \times n_{i}},$$

$$n_{rows} = 2, n_{cols} = 3$$ (3.25)

The uncertainty modeling problem therefore requires that Eqn. (3.6) be solved for $P_{11},$ $P_{12},$ $P_{21},$ and $\Delta(\delta)$ such that the nilpotency condition of Eqn. (3.20) is satisfied. As discussed in Ref. [1], block triangular matrices with nilpotent main-diagonal blocks are nilpotent. Thus, a block-triangular $P_{11}$ matrix will be determined for this example. The main-diagonal blocks of Eqn. (3.22) are nilpotent of index $\eta_i,$ i.e.: $$(P_{118} \delta_A)^{\eta_i} = 0, \quad \eta_i \leq n_i, \quad i = \alpha, x_{cg}$$ (3.26a)

where:

$$\eta_i = \text{deg}_{\text{max}} (\delta_A)$$ (3.26b)

For this example, $n_{\alpha} = 4$ and $n_{x_{cg}} = 1.$

3.2.1 Solution of $P_{21},$ $P_{12},$ and $P_{11}$ Main-Diagonal Blocks

The partitions of $P_{21},$ $P_{12},$ and the main-diagonal blocks of $P_{11}$ associated with each uncertain parameter can be computed simultaneously through appropriately defined
block Hankel matrices. The solution for each parameter is given below.

\[
\delta_\alpha \text{ Parameter:}
\]

\[
\bar{S}_{\Delta_0\delta_\alpha} = \text{Hankel}[S_{\Delta_0\delta_\alpha}, S_{\Delta_1\delta_\alpha}, S_{\Delta_2\delta_\alpha}, ... S_{\Delta_4\delta_\alpha}]
\]

\[
\bar{S}_{\Delta_1\delta_\alpha} = \text{Hankel}[S_{\Delta_1\delta_\alpha}, S_{\Delta_2\delta_\alpha}, S_{\Delta_3\delta_\alpha}, 0]
\]

where:

\[
\text{Hankel} [H_1, H_2, ..., H_n] = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ H_n & 0 & \cdots & 0 \end{bmatrix}
\]

and the sub-matrices in the above Hankel matrices are the coefficient matrices given in Eqns. (3.10), (3.12), (3.14), and (3.16). Once the Hankel matrices are formed, the matrix singular value decomposition (svd) is applied as follows

\[
\bar{S}_{\Delta_0\delta_\alpha} = U_{\delta_\alpha} \Sigma_{\delta_\alpha} V_{\delta_\alpha}^T = (U_{\delta_\alpha} \Sigma_{\delta_\alpha}^{1/2})(\Sigma_{\delta_\alpha}^{1/2} V_{\delta_\alpha}^T)
\]

\[
= P_{21\delta_\alpha} P_{12\delta_\alpha}^T
\]

where:

\[
n_{\alpha} = 7
\]

\[
\bar{P}_{21\delta_\alpha} = \begin{bmatrix} P_{21\delta_\alpha} \\ P_{21\delta_\alpha}^2 (P_{11\delta_\alpha})^2 \\ P_{21\delta_\alpha}^3 (P_{11\delta_\alpha})^3 \\ \end{bmatrix}
\]

\[
\bar{P}_{12\delta_\alpha} = P_{12\delta_\alpha} (P_{11\delta_\alpha})^2 (P_{11\delta_\alpha})^3
\]

Then:

\[
P_{21\delta_\alpha} = \begin{bmatrix} 1_2 & 0 \end{bmatrix}
\]

\[
P_{21\delta_\alpha} = \begin{bmatrix} 1_3 \\ 0 \end{bmatrix}
\]

\[
P_{11\delta_\alpha}^T = (P_{21\delta_\alpha})^T \bar{S}_{\Delta_1\delta_\alpha} (P_{12\delta_\alpha})^T
\]

where the notation \((A)^T\) designates the pseudoinverse of matrix A. Note that the \(P_{11\delta_\alpha}\) matrix is nilpotent with index 4.

\[
\delta_{xcg} \text{ Parameter:}
\]

\[
S_{\Delta_0\delta_{xcg}} = U_{\delta_{xcg}} \Sigma_{\delta_{xcg}} V_{\delta_{xcg}}^T = (U_{\delta_{xcg}} \Sigma_{\delta_{xcg}}^{1/2})(\Sigma_{\delta_{xcg}}^{1/2} V_{\delta_{xcg}}^T)
\]

\[
= P_{21\delta_{xcg}} P_{12\delta_{xcg}}^T
\]

where \(n_{xcg} = 2\) forces \(P_{11\delta_{xcg}}\) to be zero.

3.2.2 Solution of \(P_{11}\) Off-Diagonal Blocks

The \(P_{11}\) off-diagonal block was solved using the crossterms of \(S_{\delta}\), as defined by Eqn. (3.9). Since there are only two uncertain parameters, there is only one off-diagonal block of \(P_{11}\) to be solved. For the upper triangular block, the equation is given as follows.

\[
\begin{bmatrix} P_{21\delta_\alpha} \\ P_{21\delta_\alpha}^2 (P_{11\delta_\alpha})^2 \\ P_{21\delta_\alpha}^3 (P_{11\delta_\alpha})^3 \\ \end{bmatrix}
\]

\[
= \begin{bmatrix} S_{\Delta_1\delta_\alpha} \delta_{xcg} \\ S_{\Delta_2} \delta_{xcg} \\ S_{\Delta_3} \delta_{xcg} \\ S_{\Delta_4} \delta_{xcg} \end{bmatrix}
\]

Eqn. (3.35) is equivalent to a generalized linear matrix equation of the following form:

\[
AXB = C
\]

where A, B, and C are known constant matrices. In order to solve Eqn. (3.36) for the unknown matrix, X, the two
The normalized parametric solution for this example is given as follows

\[ P_{21} = \begin{bmatrix} P_{21}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{12} = \begin{bmatrix} P_{12}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11}^{\delta_{x_cg}} \\ \delta_{x_cg} \end{bmatrix} \]

The normalized parametric solution for this example is given as follows

\[ P_{21} = \begin{bmatrix} P_{21}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{12} = \begin{bmatrix} P_{12}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11}^{\delta_{x_cg}} \\ \delta_{x_cg} \end{bmatrix} \]

The normalized parametric solution for this example is given as follows

\[ P_{21} = \begin{bmatrix} P_{21}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{12} = \begin{bmatrix} P_{12}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11}^{\delta_{x_cg}} \\ \delta_{x_cg} \end{bmatrix} \]

The normalized parametric solution for this example is given as follows

\[ P_{21} = \begin{bmatrix} P_{21}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{12} = \begin{bmatrix} P_{12}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11}^{\delta_{x_cg}} \\ \delta_{x_cg} \end{bmatrix} \]

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The normalized parametric solution for this example is given as follows

\[ P_{21} = \begin{bmatrix} P_{21}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{12} = \begin{bmatrix} P_{12}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11}^{\delta_{x_cg}} \\ \delta_{x_cg} \end{bmatrix} \]

The normalized parametric solution for this example is given as follows

\[ P_{21} = \begin{bmatrix} P_{21}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{12} = \begin{bmatrix} P_{12}^{\delta} \\ \delta_{x_cg} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11}^{\delta_{x_cg}} \\ \delta_{x_cg} \end{bmatrix} \]
4. Formulation of a Validated Mixed Uncertainty Model

The LFT model developed in Sections 2 and 3 does not match the actual system (represented by the simulation model for this example) exactly. This can be attributed to many possible causes, including fit errors in the parameters in trying to limit the order of the model, and unmodeled dynamics in the form of a short period mode approximation instead of the full vehicle dynamics model. Therefore, as a final step in the modeling process, we cover the discrepancy between the simulation model and the LFT model by introducing output multiplicative uncertainties for the unmodeled dynamics and a small parameter uncertainty allowance. Using input and output time histories generated from a detailed simulation, a smallest level of unmodeled dynamics subjected to assumed levels of gust/turbulence disturbance and parameter uncertainty allowance can be computed over the flight envelope.

Figure 2 shows the block diagram used for computing model validating mixed uncertainties for the F-16 example.

\[
\begin{align*}
V & \rightarrow V_{\text{noise}} \rightarrow Z_{\Delta} \rightarrow P_{11} \cdots P_{12} \cdots P_{21} \cdots P_{22} \rightarrow 0_{2 \times 10} I_{12} 0_{2 \times 1} \rightarrow u_{\text{elev}} \\
& \quad \downarrow \Delta_\alpha \quad \downarrow \Delta_\text{q} \\
& \quad V'_{\text{dist}} \\
& \quad \downarrow \varepsilon \\
\end{align*}
\]

Figure 2. Block Diagram for Model Validation of the F-16

The two state LFT model given in the previous section is used in Figure 2. The terms, \(\delta_\Delta\) and \(\delta_{\text{q}q}\), represent the parameter variations determined in Sections 2 and 3, while \(\Delta_\alpha\) and \(\Delta_\text{q}\) denote output multiplicative uncertainties which are intended to capture the unmodeled dynamics. Discrete time measurements for a duration of about 25 seconds sampled at 10 Hertz are assumed given. The elevator command (in degrees) is generated from a uniform random signal passed through a low pass filter with a bandwidth of 2 Hertz. No measurement noise or disturbance is added for simplicity, but a small allowance for these signals, with filters given by \(V_{\text{noise}}\) and \(V_{\text{dist}}\), is included in the model validation process. All time signals used in the validation are Hanning windowed to mitigate the effects of data truncation.

The objective is to obtain a smallest model validating unmodeled dynamics at each frequency given fixed allowances in the parameter uncertainties of \(|\delta_{\Delta}| \leq 0.1\) and \(|\delta_{\text{q}q}| \leq 0.1\), plus a small noise and disturbance allowance. The simulation results from two trim cases are shown for this example. Figures 3 and 4 show the input commands and measured responses about trim conditions at \(\alpha_{\text{trim}} = 20^\circ\), \(X_{\text{cg}} = .25\) (Case 1) and \(\alpha_{\text{trim}} = 15^\circ\), \(X_{\text{cg}} = .3\) (Case 2), respectively. The time histories shown are the deviations from their respective trim values. Notice that the time responses are significantly different at the two different trim points. Since the input is not zero mean, there is a significant low frequency response of less than 0.02 Hz (Figure 4) which will not be accurately captured by the uncertainty bound identification due to the limited time record.

Figure 5 shows the smallest model validating multiplicative uncertainties \(\Delta_\alpha\) and \(\Delta_\text{q}\) for Cases 1 and 2, as well as fitted uncertainty bounds that can be used in developing weighting functions for the associated mixed uncertainty model. The respective spectrum of the measured outputs is also shown. The figures show that the multiplicative uncertainties are generally larger at higher frequencies, say beyond 1 Hertz. This is expected since rigid aircraft are less responsive at higher frequencies so that any discrepancy, which the output multiplicative uncertainties must cover, will likely be large. The uncertainty bound is also smaller for Case 1 than Case 2. This is expected since the nominal P-\(\Delta\) model corresponds to the Case 1 nominal model. The uncertainty bounds for Case 2 can be reduced by setting the \(\delta_\Delta\) and \(\delta_{\text{q}q}\) parameters to nonzero values that correspond as closely as possible to the nominal model of Case 2, and then letting the parametric uncertainty allowances (e.g., \(\pm 0.1\)) be relative to these nonzero values.

In developing a validated mixed uncertainty model that is valid over the flight envelope being considered for this example (i.e., the \(\alpha-X_{\text{cg}}\) parameter space), model validating uncertainty bounds can be identified for each trim point in the parameter space (using the appropriate nonzero \(\delta_\Delta\) and \(\delta_{\text{q}q}\) parameter values associated with each trim condition). Then, the uncertainty bound obtained for the nominal model can be modified such that coverage is obtained over all trim points in the parameter space. Alternatively, the uncertainty bounds obtained over the trim points can be parameterized over \(\alpha\) and \(X_{\text{cg}}\). If a parameterization is performed, then the LFT computation method of Section 3 can be utilized to obtain the final validated mixed uncertainty model for the system. The unmodeled dynamics part of the resulting model helps to cover the discrepancies between the "true" system and the parametric LFT model. Sources for these discrepancies include: (i) errors in the parameter fit in forming the LPV model, (ii) the differences between the LPV model and the full nonlinear coupled dynamical motion of the F-16, (iii) discrepancy in the dynamics between the two trim points, and (iv) the linear, time-invariance assumption in the parameterization of the model validating uncertainties.

5. Concluding Remarks

This paper has presented an example uncertainty modeling problem for an F-16 aircraft that illustrates the application of the modeling methods presented in Ref. [1]. A detailed nonlinear simulation was used to generate a set
of linear models for the short period mode approximation of the vehicle over variations in $\alpha$ and $x_{cg}$ that characterize nearly the full flight envelope. An LPV model was developed by parameterizing the state model matrix elements over the set of linear models. A low-order parametric LFT model was then computed based on the system LPV model. Development of a validated mixed uncertainty model was illustrated for two trim points using the parametric LFT model and a multiplicative uncertainty structure defined to characterize unmodeled dynamics.

The resulting mixed uncertainty model could be used for robust control system analysis and design using $\mu$-analysis/synthesis and LPV control methods. For example, $\mu$-analysis/synthesis could be used to design a single robust control system that provides robust stability and performance relative to variations in $\alpha$, uncertainties in $x_{cg}$, external disturbances (e.g., gusts/turbulence), sensor noise, and unmodeled dynamics. If a single robust control system cannot be designed for this problem, a robust parameter-dependent control system could be designed, using $\mu$-analysis/synthesis and LPV control methods, to provide scheduling over $\alpha$ and robustness to uncertainties in $x_{cg}$, system noise and disturbances, and unmodeled dynamics. As illustrated by the example of this paper, the uncertainty modeling methods presented in Ref. [1] provide a computer-aided uncertainty modeling capability for a broad range of practical problems.

References

Case 1: Uncertainty $\alpha$, $|\alpha|$

Case 2: Uncertainty $\alpha$, $|\alpha|$

Case 1: Uncertainty $q$, $|q|$

Case 2: Uncertainty $q$, $|q|$

Figure 5. A Smallest Model Validating Output Multiplicative Uncertainties for Cases 1 and 2 With Fitted Uncertainty Bounds for $\alpha$ (circles) and $q$ (squares), and the Spectrum of the Measured Outputs (dashed line)