PERIODIC BOUNDARY CONDITIONS FOR FINITE ELEMENT ANALYSIS OF INFINITE PHASED ARRAY ANTENNAS

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1. INTRODUCTION

The finite element method (FEM) has been used to solve for the active reflection coefficient of planar phased arrays [1]. One of the key features of the solution is a periodic boundary condition applied to the three-dimensional FEM mesh to represent periodicity in two directions (including skewed grids), thereby satisfying the Floquet condition. This paper presents the formalism for the periodicity condition, and shows some example calculations for illustrative purposes.

2. PROBLEM DESCRIPTION

Fig. 1 shows top and side views of a typical array lattice and unit cell. The radiator consists of an arbitrarily-shaped conducting structure and may include any number of dielectric regions. A cylindrical waveguide feeds the radiator through an aperture, denoted \( \Gamma_w \), in the ground plane. The unit cell is truncated at a constant-\( z \) plane, denoted \( \Gamma_R \), above the radiator's physical structure.

A typical problem is illustrated in Fig. 2, an exploded view of the tetrahedron mesh for a flared notch [2],[3] printed on a dielectric substrate. The middle group of cells represent the dielectric and the other two, extending out to the unit cell boundaries, are air. Shading identifies the substrate metallization. (This variant of the flared notch is fed from its base by a coaxial line into a coplanar waveguide section. The CPW-slotline transition is from Ho & Hart [4].)

3. SOLUTION APPROACH

The unit cell mesh is first treated as though its side walls were open-circuit boundaries. The electric field inside the volume region \( \Omega \) is expanded in vector finite elements, which are also used as testing functions for the Galerkin discretization of the weak form of the wave equation:

\[
F(\vec{E}) = \int_0^1 \left[ \frac{1}{\mu_0} \nabla \times \vec{W} \times \cdot \nabla \times \vec{E} - k_0^2 \sigma_0 \vec{W} \times \cdot \vec{E} \right] dv - jk_0 \sigma_0 \int_0^1 \vec{W} \times \cdot \sigma \times \vec{H} ds = 0
\]

(1)

Radiation conditions at \( \Gamma_w \) and \( \Gamma_R \) are imposed by substituting sums of waveguide and Floquet modes, respectively, for \( \nabla \times \vec{H} \) in the boundary term of (1) [1],[5]. This results in the matrix equation

\[
[S^L + S^W + S^R] \vec{E} = \vec{E}^{inc}
\]

(2)

where the incident field vector is due to a unit-amplitude dominant mode in the waveguide. There are no contributions to \( S^W \) from edges in \( \Gamma_R^+ \) or \( \Gamma_R^- \): Instead, edges in \( \Gamma_R^+ \) and \( \Gamma_R^- \) generate contributions that include mesh cells in adjacent unit cells (viewing the unit cell mesh in Fig. 2 as only part of an infinite mesh). This is conceptually the same as Gedney’s “overlap elements” [6].

Finally, the periodicity condition at the side walls is imposed using the matrix transformation

\[
[T] [S^L + S^W + S^R] [T]^H \vec{E}' = \vec{E}^{inc}
\]

(3)

where the prime on \( \vec{E} \) denotes that it excludes edges in \( \Gamma_{R^+} \) and \( \Gamma_{R^-} \). The superscript \( H \) denotes Hermitian. The matrix \( T \) is \( M \times N \), where \( N \) is the total number of edges and \( (N-M) \) is the number of edges in \( \Gamma_{R^+} \) and \( \Gamma_{R^-} \). It is the \( M \times M \) identity matrix augmented with the following entries:

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\[ T_{ik} = \begin{cases} \varepsilon_0 k_0 \forall x, \forall y, \\ \varepsilon_0 k_0 \sin \theta_0 \cos \phi_0, \forall x, \forall y, \\ \varepsilon_0 k_0 \sin \theta_0 \sin \phi_0, \forall x, \forall y, \end{cases} \]

where \( k_0 \) is the free space wavenumber and \( \theta_0 \) and \( \phi_0 \) are the spherical coordinate angles to which the array is scanned.

After solving for \( E' \), the active reflection coefficient may be found from the unknowns associated with waveguide aperture edges.

4. RESULTS

Fig. 3 presents computed active reflection coefficient for the flared notch radiator with \( \varepsilon_r = 6 \) for the substrate, whose height in \( z \) is 38.1 mm. The rectangular lattice has \( d_x = 36 \) mm and \( d_y = 34 \) mm. There are two blind angles in the E-plane, one due to the grating lobe onset. The other exhibits the unusual behavior of moving outward in angle with increasing frequency. This behavior, predicted by Schaubert [7], is evidently due to a guided-wave mode in the structure formed by the parallel metallized substrates.

Fig. 4 compares the hybrid finite element (HFEM) and MoM calculations of normalized active reflection coefficient for rectangular microstrip patches on two substrate thicknesses, \( 0.02 \lambda_0 \) and \( 0.08 \lambda_0 \) [8]. In each case, the lattice is square with \( d_x = d_y = 5 \lambda_0 \) and the substrate's relative permittivity is 12.8. The patch dimensions are (width, length) = (1.5 \lambda_0, 1.3 \lambda_0) and (1.5 \lambda_0, 0.98 \lambda_0) for the thin and thick cases, respectively. The slight discrepancy in the computations for the thick case is due to the difference between the idealized probe feed used in [8] and the coaxial feed used in the HFEM calculations. Aberle & Pozar [9] observed similar results when an accurate coaxial feed model was included in MoM calculations.

Further validations using the Hybrid FEM code based on the concepts described here have been performed, including arrays of open-ended waveguides, clad and unclad monopoles and printed dipoles. The same computer code was used for all cases. These results attest to the versatility of the method for predicting the scanning performance of very general phased array radiators.

REFERENCES


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**Figure 1.** Generic Array Element (left) and Lattice Geometry (right).

**Figure 2.** Exploded Unit Cell Finite Element Mesh for Printed Flared Notch.
Figure 3. Calculated Active Reflection Coefficient vs. Scan Angle for Printed Flared Notch.

Figure 4. Comparison of HFEM and MoM Calculations [7] for Microstrip Patch Arrays.