HYBRID FINITE ELEMENT/WAVEGUIDE MODE ANALYSIS FOR
INFINITE PHASED ARRAYS OF CAVITY RADIATORS

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1. INTRODUCTION

A hybrid finite element method (HFEM) has been developed for analyzing the scanning properties of cavity radiators imbedded in an infinite phased array. By expressing the transverse boundary fields in terms of waveguide modes on the cavity's feed side and Floquet modes on the radiating side, continuity conditions are imposed on the variational functional. Three dimensional vector finite elements are used to expand the electric field inside the cavity. This paper outlines the approach and presents validation results for its implementation in a computer code. Those results demonstrate that the method is accurate and versatile, correctly predicting the scanning performance of a large variety of practical array radiators.

2. APPROACH

The generic problem is shown in Figs. 1 and 2, with the region \( \Omega \) denoting a cavity that is bounded by a surface \( \Gamma \) comprised of perfect conductors and two apertures \( \Gamma_F \) and \( \Gamma_R \) which are the feed and radiating aperture boundaries, respectively. The cavity may contain an arbitrary distribution of linear, isotropic materials and conducting obstacles. The radiating apertures are flush with the surface of an infinite ground plane. The variational statement (see, for example (1)) is the weak form of the Helmholtz equation for the time-harmonic electric field using a trial function \( \mathbf{W} \):

\[
F(\mathbf{E}) = \int_{\Omega} \left\{ \frac{1}{\mu_r} \mathbf{\nabla} \times \mathbf{W}^* \cdot \mathbf{\nabla} \times \mathbf{E} - \kappa_0^2 \varepsilon_0 \mathbf{W}^* \cdot \mathbf{E} \right\} dv - j \kappa_0 \varepsilon_0 \int_{\Gamma} \mathbf{W}^* \cdot \mathbf{n} \times \mathbf{H} ds = 0
\]  

(1)

The second integral is an expression for power flow across \( \Gamma \), which is nonzero only over the apertures \( \Gamma_F \) and \( \Gamma_R \). The transverse field in the feed aperture may be written in terms of orthonormal mode functions of the waveguide [2,3], denoted \( \mathbf{g}_i \) whose modal admittances are \( Y_i \):

\[
\mathbf{a} \times \mathbf{H}_f \bigg|_{z=0} = \mathbf{V}_0 \mathbf{g}_0 + \sum_{i=0}^{\infty} C_i Y_i \mathbf{g}_i
\]  

(2)

The first term on the right represents the incident field of unit amplitude carried by the dominant mode. The index \( i \) includes modes \( m,n \) and both TE and TM. After solving for the interior field, the unknown coefficients \( C_i \) may be found from

\[
C_i = \int_{\Gamma} \mathbf{E}_i \cdot \mathbf{g}_i ds - \delta_{0i}
\]  

(3)

where \( \delta_{0i} \) is the Kronecker delta. \( C_0 \) is the reflection coefficient observed in the waveguide at \( z<0 \). Similarly, the radiating aperture fields may be written in terms of orthonormal "Floquet modes" [4,41-42].

The electric field is expanded in terms of linear finite elements [1] over a mesh of tetrahedra (cells) subdividing \( \Omega \):
\[ \mathbf{E} = \sum_{s=1}^{N} c_s \mathbf{\tilde{v}}_s(x,y,z) \]  

(4)

with \( s \) an edge index, \( c_s \) a complex scalar coefficient and \( \mathbf{\tilde{v}}_s \) a vector function that is zero outside those cells adjoining edge \( s \). Using Galerkin's method, the functional is recast as the matrix equation

\[ \left[ S^I + S^F + S^R \right] \mathbf{E} = \mathbf{E}^{\text{inc}} \]  

(5)

with the interior terms given by

\[ S^I_{st} = \int \mathbf{\nabla} \times \mathbf{\nabla} \cdot \mathbf{\nabla} \times \mathbf{\tilde{v}}_s \cdot \mathbf{\tilde{v}}_t \, dv - k_0^2 \int c_s \mathbf{\tilde{v}}_s \cdot \mathbf{\tilde{v}}_t \, dv \]  

(6)

This matrix is \( N \times N \) and sparse. The feed aperture submatrix and right-side terms are

\[ S^F_{st} = jk_0 \theta_0 \sum_{i=0}^{\infty} y_i \Psi_{st} \Psi_{st} \]  

(7)

\[ E_s^{\text{inc}} = 2jk_0 \theta_0 v_0 \Psi_{s0} \]  

(8)

\[ \Psi_{st} = \int \mathbf{\nabla} \cdot \mathbf{\tilde{v}}_s \cdot \mathbf{\tilde{t}}_i \, ds \]  

(9)

Edges are indexed starting with the feed aperture and ending at the radiating aperture. Consequently, \( S^F \) is zero except for its upper left corner, which is dense. Similarly, \( S^R \) is zero except its lower right corner, also dense. Its entries are as follows, resulting from summing TE and TM modes, and given in terms of \( k_{\text{mn}} \), \( k_{\text{ym}} \), the Floquet harmonics appropriate to a skewed lattice [4:311], and scanning to the angle \((\theta_0, \phi_0)\) in spherical coordinates:

\[ S^R_{st} = \int_{\partial R} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{r_m} \mathbf{\tilde{t}}_{s,m,n} \mathbf{\tilde{t}}_{t,m,n} \begin{bmatrix} k^2 - k_{\text{mn}}^2 & k_{\text{mn}} k_{\text{ym}} \\ k_{\text{mn}} k_{\text{ym}} & k^2 - k_{\text{ym}}^2 \end{bmatrix} \mathbf{\tilde{v}}_{s,m,n} \]  

(10)

\[ \mathbf{\tilde{t}}_{s,m,n} = \int_{\partial R} \mathbf{\tilde{v}}_s \mathbf{e}^{j(k_{\text{mn}} r + k_{\text{ym}} \gamma)} \, ds \]  

(11)

\[ k_{\text{mn}} = k^2 - k_{\text{mn}}^2 - k_{\text{ym}}^2 \]  

(12)

\[ k_{\text{ym}} = \frac{2\pi m}{d_x} - k_0 \sin \theta_0 \cos \phi_0 \]  

(13)

The system of equations is solved by either LU decomposition or the conjugate gradient method, giving the vector \( \mathbf{E} \) of coefficients \( c_s \). From these, we compute the reflection coefficient in the waveguide, and the element gain from the transmission into co- and cross-polarized components of the 0,0 Floquet mode.

3. REPRESENTATIVE RESULTS

Fig. 3 shows the calculated active element gain vs. scan angle for an array of rectangular waveguides, arranged in a triangular lattice, with and without conducting irises. The HFEM results compare well to the multimode calculations from Lee and Jones [5]. The iris blocks the left and
right ¼ of each aperture. Fig. 4 shows results for a circular waveguide array, also with a triangular lattice, with and without dielectric plugs inside the radiating apertures. The HFEM calculations are compared to data (also multimode calculations) given by Amitay et al. [4]. Both figures reflect results using waveguide modes \( m, n \) up to 4, 3 and Floquet modes \(-5 \leq m, n \leq 5\). In each case the finite element model was a very short waveguide section, only two cells deep (3 for the dielectric plug case) in the \( z \)-dimension and approximately 10 edges per wavelength in \( x \) and \( y \).

4. CONCLUSIONS
A hybrid computational method using vector finite elements in conjunction with waveguide and Floquet modes was developed and demonstrated. The validation results show that the method is versatile enough to handle a wide range of practical phased array radiators. It is also capable of modeling radiators with irregular and inhomogeneous dielectric fillings, a capability that is difficult or impossible to achieve by other methods.

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REFERENCES
Figure 3. Active Element Gain vs. H-Plane Scan Angle, Triangular Lattice of Rectangular Waveguides with & without Conducting Iris: width $a = 0.905\lambda_o$, height $b = 0.4\lambda_o$, $d_x = 1.008\lambda_o$, $d_y = 0.504\lambda_o$, $\gamma = 45^\circ$. Curves are from Lee and Jones [5].

Figure 4. Active Reflection Coefficient vs. E-Plane Scan Angle, Triangular Lattice of Circular Waveguide Radiators with & without Dielectric Plug: radius $a = 0.34\lambda_o$, $d_x = 0.714\lambda_o$, $d_y = 0.618\lambda_o$, $\gamma = 60^\circ$, $\epsilon_r = 1.8$, and plug depth = $0.429\lambda_o$. Curves are from Amitay, Galindo & Wu [4:293].