STABILITY RESULTS APPLICABLE TO ADAPTIVE CONTROL

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ABSTRACT

We prove some relatively simple stability results for a particular system of time-varying difference equations. We then show that these results are useful in analyzing the stability and convergence of a wide class of adaptive control schemes. In particular, we apply the results to demonstrate the stability and convergence of two quite disparate adaptive control algorithms.

1. INTRODUCTION

Two important, practical issues in the design of any adaptive control system are the convergence of the adaptive control algorithm and the stability of the overall closed-loop system. In particular, the desired control objective, such as tracking or pole-assignment, should be realized asymptotically and, at the same time, all system variables must remain bounded. Considerable research effort has been devoted to these issues in recent years, and several different approaches have evolved for studying them (see, for example, [1]-[3]).

In this paper, we present some simple results which appear to be quite useful in the convergence and stability analysis of a number of seemingly disparate discrete-time adaptive control schemes. The results are basically stability theorems for a forced linear time-varying system of difference equations whose forcing function satisfies a certain constraint defined by states of the system. In Section 2, we state the results and prove them. Then in Section 3, we demonstrate their applicability to adaptive control. For this purpose, we consider in detail two fundamentally different adaptive control schemes that have been analyzed previously by other means. The first is a self-tuning controller for tracking based upon the one-step-ahead optimal control criterion [4]. The second scheme is a regulator design that utilizes an adaptive observer [5]. For both schemes, we show that the convergence and stability issues reduce to the analysis of a system of difference equations to which our stability results apply.

2. STABILITY RESULTS

The time-varying system of difference equations that is of interest to us takes the form

\[ x(t+1) = F(t)x(t) + v(t) \quad t \geq 0, \]  

where \( x(t) \) and \( v(t) \) are real vectors of finite dimension, and where the growth of the forcing term \( v(t) \) is related to the growth of \( x(t) \) (and possibly past values, \( x(t-1), \ldots, x(t-N) \)) in a fashion to be defined below.

Our first result is embodied in the following theorem.

Theorem 2.1: Consider the time-varying difference equation (2.1) and suppose that the following conditions hold:

C1: The sequence of matrices \( \{F(t)\} \) and the initial state \( x(0) \) are bounded.

C2: The free system

\[ z(t+1) = F(t)z(t) \]  

is exponentially stable.

C3: There exist sequences \( \{\gamma(t)\} \) and \( \{\delta(t)\} \) such that \( 0 \leq \gamma(t) \leq \delta(t) \leq \Delta < \infty \) and

\[ |v(t)| \leq \gamma(t) |x(t)| + \delta(t), \]  

where \(|-|\) denotes the Euclidean norm. Under these conditions, if \( |\gamma(t)| < 0 \), then \( x(t) \) and \( v(t) \) are bounded. If, in addition, \( \delta(t) = 0 \), then \( x(t) \) and \( v(t) \) are bounded.

Proof: Let \( \Phi(\cdot,\cdot) \) be the transition matrix for the system (2.2). By Condition C2, there exist constants \( \mu > 0 \) and \( 0 < \lambda < 1 \) such that

\[ |\Phi(t,t')| \leq \mu |t-t'| \quad \text{for all } t \geq 0 \text{ and } t \geq t'. \]
\( \varepsilon > 0 \) be arbitrarily small and such that with \( \lambda(c) = \lambda + \varepsilon \mu \) we have \( 0 \leq \lambda(c) < 1 \).

Suppose now that \( \{Y(t)\} \to 0 \). Then there exists an integer \( t_0 \) depending upon \( \varepsilon \) such that \( Y(t) \leq \varepsilon \) and \( \delta(t) \leq \varepsilon \) for all \( t \geq t_0 \), where \( \varepsilon(c) = \varepsilon \) if \( \delta(t) \to 0 \) and \( \varepsilon(c) = \Delta \) otherwise. Condition C1 ensures that \( \{x(t)\} \) is bounded. It remains to show that \( \{x(t)\} \) is bounded. Now equation (2.1) implies that

\[
x(t) = \sum_{j=0}^{t-1} \phi(t,j+1) v(j) t \Delta t_0.
\]  

(2.4)

Taking norms, we have

\[
|x(t)| \leq \sum_{j=0}^{t-1} |v(j)|, \quad \lambda c \sum_{j=0}^{t-1} |v(j)| \Delta t_0.
\]  

(2.5)

Then using the constraint (2.3) and the bounds upon \( Y(t) \) and \( \delta(t) \), we obtain

\[
|x(t)| \leq M(t) + \sum_{j=0}^{t-1} |v(j)|, \quad \lambda c \sum_{j=0}^{t-1} |v(j)| \Delta t_0.
\]  

(2.6)

where

\[
M(t) = \sum_{j=0}^{t-1} |v(j)| \lambda^{-j} \Delta t_0.
\]  

(2.7)

Note that the inequality (2.6) is implicit in \( \{x(t)\} \), but we desire an explicit bound on \( \{x(t)\} \). To this end, we write (2.7) as

\[
|x(t)| \leq \sum_{j=0}^{t-1} |v(j)| \lambda^{-j} \Delta t_0.
\]  

(2.8)

and then apply the discrete Gronwall-Bellman Lemma (see Appendix) with \( f(t) = |x(t)| \lambda^{-t}, g(t) = M(t) \lambda^{-t} \) and \( h(t) = \varepsilon \mu / \lambda \). The resulting inequality is, after some simplification,

\[
|x(t)| \leq \sum_{j=0}^{t-1} |v(j)| \lambda^{-j} \Delta t_0.
\]  

(2.9)

From (2.7) we note that \( M(t) \) is a monotone decreasing sequence, and thus we obtain from (2.9)

\[
|X(t)| \leq M(t_0) c \mu (1-\lambda(c)^{-1}) t \Delta t_0.
\]  

(2.10)

Since \( M(t) \) is bounded, we conclude from (2.10) that \( \{x(t)\} \) is bounded. That \( \{v(t)\} \) is also bounded follows from (2.3) and the boundedness of \( \{Y(t)\} \) and \( \{\delta(t)\} \).

Now suppose that it is also true that \( \{\delta(t)\} \to 0 \). Then there exists an integer \( K \geq t_0 \) such that \( \mu^t \Delta t_0 \frac{|x(t)|}{c} \leq \varepsilon \) for all \( t \geq K \). We may then obtain from (2.10) and (2.7) that

\[
|x(t)| \leq M(t_0) e \mu (1-\lambda(c)^{-1}) t \Delta t_0.
\]  

(2.11)

But since \( \varepsilon \) is arbitrarily small, we conclude that \( \{x(t)\} \to 0 \). It then follows that \( \{v(t)\} \to 0 \) also, and we have proven the theorem.

This stability result is merely a special case of our second theorem, in which the constraint on \( \{v(t)\} \) is generalized to include past values of \( x(t) \).

Theorem 2.2: Consider the time-varying difference equation (2.1) and suppose C1 and C2 hold. Further, suppose that the following condition holds:

C4: There exist an integer \( N \geq 0 \) and sequences \( \{Y(t)\} \) and \( \{\delta(t)\} \) such that \( 0 \leq \delta(t) \leq \varepsilon \) and

\[
|\delta(t)| \leq Y(t) \sum_{i=0}^{N} |x(t-i)| + \delta(t).
\]  

(2.12)

Under these conditions, if \( \{Y(t)\} \to 0 \), then \( \{x(t)\} \) and \( \{v(t)\} \) are bounded. Moreover, if \( \{\delta(t)\} \to 0 \), then \( \{x(t)\} \to 0 \) and \( \{v(t)\} \to 0 \).

Proof: The proof makes use of Theorem 2.1. Let

\[
\bar{x}(t) = \begin{bmatrix} x(t) \\ \vdots \\ x(t-N) \end{bmatrix}.
\]  

(2.13)

Then from (2.1) we obtain

\[
\bar{x}(t+1) = F(t) x(t) + \bar{v}(t)
\]  

(2.14)

where
Suppose that the process to be controlled is a discrete-time single-input, single-output system represented by a difference equation of the form

\[ a(q^{-1})y(t) = q^{-d}b(q^{-1})u(t) \quad t \geq 0, \]  

(3.1)

where \( a(q^{-1}) \) and \( b(q^{-1}) \) are polynomials in the backward shift operator \( q^{-1} \) defined by

\[ a(q^{-1}) = a_0 + a_1 q^{-1} + \ldots + a_n q^{-(n+1)} \quad a_0 = 1 \]  

(3.2)

\[ b(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_m q^{-m} \quad b_0 \neq 0. \]  

(3.3)

The sequences \( \{y(t)\} \) and \( \{u(t)\} \) are the output and input sequences, respectively, and \( d \geq 1 \) represents the pure time delay. It is assumed that the integers \( n, m \) and \( d \) are known yet the parameters \( \{a_i\} \) and \( \{b_i\} \) are unknown but constant. Let \( \{y^*(t)\} \) be a given, bounded reference sequence. The objective of the control is to cause the output sequence \( \{y(t)\} \) to track \( \{y^*(t)\} \) and to do so in a manner which insures that \( |y(t)| \) and \( |u(t)| \) remain bounded.

3.1. A SELF-TUNING DESIGN

The first approach to the problem that we shall analyze is a scheme proposed in [4]. Motivated by a dead-beat or one-step ahead control criterion, this approach adapts or tunes the control law directly. First, we shall review the scheme, and then we shall employ Theorem 2.2 to prove that it is globally stable and convergent.

By successive substitution, it is easy to show that (3.1) may be put into the form of a d-step ahead predictor

\[ y(t+d) = a(q^{-1})y(t) + b(q^{-1})u(t) \quad t \geq 0, \]  

(3.4)

with

\[ a(q^{-1}) = a_0 + a_1 q^{-1} + \ldots + a_{n-1} q^{-(n+1)} \]  

(3.5)

\[ b(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_m q^{-m} \]  

(3.6)

where \( \{a_i\} \) and \( \{b_i\} \) are functions of the unknown parameters \( \{a_i\} \) and \( \{b_i\} \). Letting

\[ e(t+d) = y^*(t+d) - y(t+d) \]  

(3.7)

and using (3.4) - (3.6) we find that
where $\mathbf{e}$ and $\mathbf{y}(t)$ are $(n+m+d)$-vectors defined by

$$\Theta = (1, -\alpha_0, \ldots, -\alpha_{n-1}, -\beta_1, \ldots, -\beta_{m+d-1})^T$$

$$\phi(t) = (y^*(t+d), y(t), \ldots, y(t-n+t), u(t-1), \ldots, u(t-m-d+1))^T$$

If $\Theta$ were known, then the tracking error $e(t+d)$ could be made identically zero by setting $u(t) = \Theta^T \phi(t)$. This suggests, therefore, an adaptive control law of the form

$$u(t) = \hat{\Theta}(t)^T \phi(t).$$

The sequence $\{\hat{\Theta}(t)\}$ is then computed recursively by the following algorithm:

$$\hat{\Theta}(t) = \hat{\Theta}(t-d) + \rho e(t) \phi(t-d)[1 + \left|\phi(t-d)\right|^2]^{-1}$$

with

$$0 < \rho \leq 2$$

and $\{\theta(t); t = 0, 1, \ldots, d-1\}$ given. Note that this is a direct scheme in the sense that the controller gains are computed directly. This adaptive controller has the following property.

**Lemma 3.1:** If the adaptive control defined by (3.11) - (3.13) is applied to (3.1), then the following conditions apply:

1. $\left|\hat{\Theta}(t+d) - \Theta^T \phi(t)\right| = 0$ for all $t \geq 0$

2. $\lim_{t \to \infty} \frac{e(t+d)^2}{1 + \left|\phi(t)\right|^2} = 0$.

**Proof:** The result follows from standard arguments found, for example, in [4] and [6].

We may now show this scheme to be both stable and convergent using Theorem 2.2.

**Theorem 3.1:** Suppose that $\{|y^*(t)|\}$ is bounded and that all the zeros of $b(q^{-1})$ lie strictly inside the unit disc. Under these conditions, if the adaptive control law (3.11) - (3.13) is applied to the process (3.1), then $\{|y(t)|\}$ and $\{|u(t)|\}$ are bounded and

$$\lim_{t \to \infty} [y^*(t) - y(t)] = 0.$$

**Proof:** Let $x(t)$ be the $(n+m+d-1)$ vector defined by

$$x(t) = \begin{bmatrix} y(t) \\ \vdots \\ y(t-n+1) \\ u(t-1) \\ \vdots \\ u(t-m-d+1) \end{bmatrix}$$

Then (3.1) - (3.3) imply that

$$x(t+1) = Fx(t) + v(t),$$

with

$$F = \begin{bmatrix} S_T & 0 \\ -\Theta & -\Theta^T \phi(t-d) \end{bmatrix} S_{m+d-1}^{-1} e_1 b_1^T$$

and

$$v(t) = \begin{bmatrix} y(t+1) \\ \vdots \\ y(t+d-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\Theta$ is the $(n+m+d-1)$ vector $(b_1, \ldots, b_{m+d-1})^T$, $e_1$ is the $(m+d-1)$ vector $(1, 0, \ldots, 0)^T$, $b_1^T b_1 / b_0$, $a_1 = a_1 / b_0$ and $S_k$ is the $k \times k$ matrix defined by
We shall now show that the system (3.16) satisfies the conditions of Theorem 2.2. First, note that the only nonzero eigenvalues of $F$ must be zeros of $b(q^{-1})$. Therefore, since all the zeros of $b(q^{-1})$ lie strictly inside the unit disc, the system $z(t+1) = Fz(t)$ is exponentially stable. Next, we must establish that Condition C4 holds. From (3.18), it follows that there exists an $a > 0$, depending upon $|a_1|$ such that for $N \geq \max \{d-1, n\}$

$$|v(t)| \leq a \sum_{i=0}^{N-1} |y(t-d+i)|. \quad (3.20)$$

Let

$$\varepsilon(t) = |e(t)| \left[1 + |\phi(t-d)|^2 \right]^{-1/2} \quad (3.21)$$

and note that

$$|e(t)| = \varepsilon(t) \left[1 + |\phi(t-d)|^2 \right]^{-1/2} \varepsilon(t) \left[1 + |\phi(t-d)| \right]. \quad (3.22)$$

Since $|y*(t)|$ is bounded, there exists a constant $M < \infty$ such that $|y*(t)| \leq M$ for all $t$. Further, since $|\phi(t)| = \sqrt{y*(t)T(x(t))}$, we have

$$|\phi(t-d)| \leq M + \|x(t-d)\|. \quad (3.23)$$

Thus, we obtain from (3.7), (3.22) and (3.23), that

$$|y(t)||y*(t)| + |e(t)| \leq M + \varepsilon(t)(1+M) + \varepsilon(t)||x(t-d)||. \quad (3.24)$$

Applying this result to (3.20), we obtain

$$|v(t)| \leq Y(t) \sum_{i=0}^{N-1} |x(t-i)| + \delta(t), \quad (3.25)$$

where

$$Y(t) = \max_{i=0}^{N} |x(t-d+i)|, \quad (3.26)$$

and

$$\delta(t) = a \sum_{i=0}^{N-1} \varepsilon(t)(i+M). \quad (3.27)$$

Note that Lemma 3.1(ii) and (3.21) imply that $|y(t)| \leq 0$. Therefore $\{y(t)\} \rightarrow 0$ and $\{\delta(t)\}$ is bounded. The hypotheses of Theorem 2.2 being thus fulfilled, we conclude that $\{x(t)\}$ is bounded and, consequently, $\{y(t)\}$ and $\{u(t)\}$ are bounded. Since $\{\delta(t)\}$ is bounded, it follows from Lemma 3.1(ii) that $|e(t)| \rightarrow 0$. Thus the theorem is proven.

This result was first stated in [4]. There the principal tool suggested for the proof was the so-called "key technical lemma," which has subsequently been shown to be an effective tool for the stability analysis of a whole class of adaptive control schemes [2]. Our approach here, using Theorem 2.2 gives an alternative way of analyzing this class of systems, and is in a certain sense somewhat more intuitive, direct and natural.

3.2. A REGULATOR DESIGN UTILIZING AN ADAPTIVE OBSERVER

The second adaptive control scheme which we shall consider is one which relies upon an adaptive observer. Such an approach has been considered, for example, in [5], [7], [8]. To simplify our treatment, we shall look only at the case when output regulation is the objective; that is, we take $y*(t) \rightarrow 0$ for all $t \geq 0$. As before, we first review the scheme. Then we will apply Theorem 2.1 to analyze its stability and convergence properties.

Let the process to be controlled be represented by (3.1) with $d = 1$ and $m = n-1$. We may then rewrite (3.1) as

$$y(t) = \delta^T \phi(t-1), \quad (3.28)$$

where

$$\delta(t) = a \sum_{i=0}^{N-1} M + \varepsilon(t)(i+M). \quad (3.29)$$

and

$$\varepsilon(t) = |e(t)| \left[1 + |\phi(t-d)|^2 \right]^{-1/2} \varepsilon(t) \left[1 + |\phi(t-d)| \right]. \quad (3.30)$$

and

$$|e(t)| \leq Y(t) \sum_{i=0}^{N-1} |x(t-i)| + \delta(t), \quad (3.31)$$

where

$$Y(t) = \max_{i=0}^{N} |x(t-d+i)|. \quad (3.32)$$

and

$$\delta(t) = a \sum_{i=0}^{N-1} \varepsilon(t)(i+M). \quad (3.33)$$

Note that Lemma 3.1(ii) and (3.21) imply that $|y(t)| \leq 0$. Therefore $\{y(t)\} \rightarrow 0$ and $\{\delta(t)\}$ is bounded. The hypotheses of Theorem 2.2 being thus fulfilled, we conclude that $\{x(t)\}$ is bounded and, consequently, $\{y(t)\}$ and $\{u(t)\}$ are bounded. Since $\{\delta(t)\}$ is bounded, it follows from Lemma 3.1(ii) that $|e(t)| \rightarrow 0$. Thus the theorem is proven.

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\[ \theta = (-a_1, \ldots, -a_n, b_0, \ldots, b_{n-1})^T \] (3.29)

\[ \phi(t-1) = (y(t-1), \ldots, y(t-n), u(t-1), \ldots, u(t-n))^T \] (3.30)

The process (3.28) may also be written in observer-canonical state form

\[ z(t+1) = A(a)z(t) + bu(t) \quad z(0) = z_0 \] (3.31)

\[ y(t) = cz(t), \] (3.32)

where \( A(a) \) is an \( n \times n \) matrix defined by

\[ A(a) = S_n^T \] (3.33)

with

\[ c = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \] (3.34)

The initial state \( z_0 \) is a function of \( \phi(0) \). In fact, it is easy to show (cf. [8]) that the state \( z(t) \) is related to \( \phi(t-1) \) by

\[ z(t) = H(\theta)\phi(t-1) \quad t \geq 1, \] (3.35)

where \( H(\theta) \) is the \( n \times 2n \) matrix of the form

\[ H(\theta) = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_n & b_0 & b_1 & \ldots & b_{n-1} \\ -a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_n & \ldots & 0 & b_{n-1} \end{bmatrix} \] (3.36)

Since the parameter vector \( \theta \) is not known, we define a predictor of form similar to (3.28):

\[ \hat{y}(t) = \hat{\theta}(t-1)^T \phi(t-1), \] (3.37)

where \( \hat{\theta}(t) = (-\hat{a}(t)^T, \hat{b}(t)^T)^T \) is some estimate of \( \theta \) derived from \( u(k): k=0, \ldots, t-1 \) and \( y(k): k=0, \ldots, t \). Then, following the lead suggested in [8], we define the estimate \( z(t) \) by

\[ z(t) = H(\hat{\theta}(t-1))\phi(t-1). \] (3.38)

We can easily show the following result.

Lemma 3.2: The sequence \( \{\hat{z}(t)\} \) defined by (3.38) satisfies

\[ \hat{z}(t+1) = A(\hat{a}(t))\hat{z}(t) + \hat{b}(t)u(t) + \hat{\alpha}(t)\hat{e}(t) + \sum_{j=n}^{\infty} H(j\hat{\theta}(t))\phi(t-1) \] (3.39)

\[ \hat{y}(t) = cz(t) \] (3.40)

where

\[ \hat{e}(t) = \hat{y}(t) - y(t) \] (3.41)

\[ \hat{\alpha}(t) = \hat{\theta}(t) - \hat{\theta}(t-1). \] (3.42)

Proof: See [8] or [9].

Regulation is then achieved by computing a feedback control

\[ u(t) = K(\hat{\theta})z(t). \] (3.43)

The stability and convergence of this approach relies upon two basic assumptions.

Assumption 3.1: The parameter estimate sequence \( \{\hat{\theta}(t)\} \) is generated so that the following properties hold:

1. \( \{\hat{\theta}(t)\} \) is bounded
2. \( \lim_{t \to \infty} \|\hat{\theta}(t) - \hat{\theta}(t-1)\| = 0 \)
3. There exist non-negative sequences \( \{a(t)\} \) and \( \{b(t)\} \) which converge to zero such that

\[ |\hat{y}(t) - y(t)| \leq a(t) \|\hat{\theta}(t-1)\| + b(t). \] (3.44)
Assumption 3.2: The sequence of feedback matrices \( \{K(t)\} \) is bounded and such that the system

\[
\ddot{z}(t+1) = [A(\hat{a}(t)) + b(\hat{t})K(t)] \ddot{z}(t)
\]  

(3.45)

is exponentially stable.

Assumption 3.1 holds for most standard parameter estimation algorithms, including recursive least squares and gradient algorithms (cf. [2], [5]). Control laws satisfying Assumption 3.2 have been realized based upon the LQ regulator solution [5] and a pole-placement approach [5].

Now using Theorem 2.1, it is easy to establish the following stability and convergence properties for adaptive controller outlined above.

Theorem 3.2: Suppose that Assumptions 3.1 and 3.2 hold and that the adaptive control scheme defined by (3.37), (3.38) and (3.43) is applied to the process (3.28). Then the sequences \( \{y(t)\}, \{u(t)\} \) and \( \{z(t)\} \) remain bounded and \( \{y(t)\} \to 0 \).

Proof: Let

\[
x(t) = \begin{bmatrix} \ddot{z}(t) \\ \dot{\phi}(t-1) \end{bmatrix}
\]  

(3.46)

Then (3.39) - (3.43) imply that

\[
x(t+1) = F(t) \cdot x(t) + v(t)
\]  

(3.47)

where

\[
F(t) = \begin{bmatrix} A(\hat{a}(t)) + b(\hat{t})K(t) & 0 \\ e_1 \cdot c + e_{n+1} \cdot K(t) & F_1 \end{bmatrix}
\]  

(3.48)

and

\[
v(t) = \begin{bmatrix} \hat{a}(t) \\ \{y(t) - y(\hat{t})\} \end{bmatrix} + S_n \cdot H(\hat{\Theta}(t)) \cdot \phi(t-1),
\]  

(3.49)

\[
F_1 = \begin{bmatrix} S_n^T & 0 \\ 0 & S_n^T \end{bmatrix}
\]  

(3.50)

and where \( e_1 \) is an \( 2n \)-vector whose only nonzero component is a unity \( i \)-th component. By Assumption 3.2, the free system \( x(t+1) = F(t) \cdot x(t) \) is clearly exponentially stable. Also, from (3.49) and (3.44) we find that

\[
\|v(t)\| \leq \|S_n H(\hat{\Theta}(t))\| \cdot \|\Delta(\hat{\Theta}(t))\| \cdot \|\phi(t-1)\| + \delta(t),
\]  

(3.51)

where

\[
\|F(t)\| = \|S_n H(\hat{\Theta}(t))\| \cdot \|\Delta(\hat{\Theta}(t))\| \cdot \|\phi(t-1)\|.
\]  

(3.52)

Invoking Assumption 3.1, we easily deduce that \( \{V(t)\} \) and \( \{\delta(t)\} \) converge to zero. Hence, by Theorem 2.1, \( \{\hat{\theta}(t)\} \) converges to zero and the conclusions of the theorem follow.

4. CONCLUDING REMARKS

We have stated and proved stability results for a particular time-varying system of difference equations. We have then illustrated how these results may be employed to investigate the convergence and stability of adaptive control systems. In particular, we have studied in some detail two adaptive control schemes which were founded upon different design philosophies. In each case, our technique was the same. First, we set up a difference equation (2.1) where the state included input and output data of the process under control. Second, we verified that the corresponding unperturbed or free system (2.2) was exponentially stable. For each scheme, this property was insured by the way in which the control law was designed. Third, we established that the forcing term \( v(t) \) satisfied the constraint (2.3) or, more generally, (2.12). The sequences \( \{Y(t)\} \) and \( \{\delta(t)\} \) were shown to meet the required boundedness conditions as a result of the manner in which the parameter estimator was designed (cf. Lemma 3.1 and Assumption 3.1). Finally, we applied Theorem 2.1 or 2.2 to establish the convergence and stability properties of the adaptive control system.

While we have considered only two adaptive schemes, we must stress that a system such as (2.1) with constraint (2.12) underlies a number of other schemes. Thus the same technique applies in principle to these schemes as well.
REFERENCES


APPENDIX

The following lemma is used to prove the stability results in Section 2. It is in essence a discrete version of the Gronwall-Bellman Lemma which finds wide application in the mathematical analysis literature. It provides an effective means for transcribing a functional inequality implicit in a sequence \( f(t) \) into an explicit bound on \( f(t) \).

**Lemma:** Let \( f(t) \), \( g(t) \) and \( h(t) \) be sequences of real numbers with \( h(t) \geq 0 \) for all integers \( t \geq t_0 \). Furthermore, suppose that the inequality

\[
f(t) \leq g(t) + \sum_{j=t_0}^{t-1} f(j) h(j)
\]

holds for all \( t \geq t_0 \). Then, the inequality

\[
f(t) \leq g(t) + \sum_{j=t_0}^{t-1} g(j) h(j) \leq (1+h(t)) \sum_{j=t_0}^{t-1} g(j) h(j)
\]

also holds for all \( t \geq t_0 \). (By convention, \( \sum_{i=0}^{0} a_i = 0 \) and \( \Pi (\cdot) = 1 \).)

**Proof:** See [9].