Air Vehicle Optimal Trajectories Between Two Radars

Michael C. Novy
Air Force Research Laboratory
1864 Fourth Street, Bldg 15
Wright-Patterson AFB, OH 45432

David R. Jacques
Meir Pachter
Air Force Institute of Technology
2950 P Street, Bldg 640
Wright-Patterson AFB, OH 45433

Abstract
The problem of formulating and analyzing the single vehicle path planning problem for radar exposure minimization is addressed. A single vehicle with given initial and final positions is exposed to two radars and the optimal path between the radars is sought. The objective cost of the optimal paths are compared with the direct path (a straight line) as well as trajectories generated using the graphical Voronoi path planning approach. Finally, each radar is given a different weight, simulating differing transmission powers, and optimal paths are sought for the same radar configurations. The objective costs of these trajectories are again compared to the direct path and the weighted Voronoi path. The nonlinear differential equations governing the optimal trajectory against multiple radars constitute a difficult, numerically sensitive two-point boundary value problem. Results indicate that approaching the Voronoi-generated curves in an optimal way from the endpoints may provide for feasible on-line and real-time utilization.

1 Introduction
Optimal path planning research to date has been extensive, and many techniques exist to formulate and solve the problem. A common thread through much of the literature is that the formulations often include complicated dynamics and multiple constraints. This results in complex problems whose solution can often only be found utilizing numerical techniques. By approaching a simple problem first and understanding its solution, insights can be gained which can later be applied to more complicated problems. This research aims to explore the feasibility of using geometric, deterministic solutions which, while suboptimal, may approximate the optimal solution to within acceptable limits.

This endeavor builds upon the work of Pachter, investigating the problem of minimizing the exposure of an air vehicle to a single radar. This initial work was extended by Hebert to handle multiple aircraft performing a coordinated rendezvous and a single aircraft with two radars. While an analytic solution was found for certain scenarios of the single radar problem, the general two-radar problem was solved numerically using a shooting method. The specific objective here is to analyze the optimal paths traveling between two radars for several geometrically symmetric scenarios and compare the resultant optimal paths against the direct (i.e. straight line) path and a path following the locus of equal radar power, the Voronoi path. Judd and McLain also considered a splined approach based on the Voronoi edges, but they did not consider the case of unequal power radars and no comparisons were made between the splined paths and the "optimal" paths. For a given radar geometry and power ratios, the Voronoi edge can be found deterministically, and it is straightforward to generate entrance (exit) paths that get you onto (off of) the Voronoi edge as necessary. The attraction of this approach is that it does not require the solution of a difficult optimization problem.

This research is limited to exploring the constant-velocity, single vehicle, constant radar cross section scenario. The goal here is to understand the fundamental problem so that when additional constraints are prescribed, the insights gained from this problem can be applied and perhaps an easier and better approach to
the optimal solution can be found.

2 Problem Formulation

2.1 Continuous Performance Index

The amount of power received by a radar is given by the radar range equation \[ P_r = \frac{P_t G A_o \sigma}{(4\pi)^2 R^2} \] (1),

where \( P_t \) is the power of the radar transmitter, \( G \) is the transmitting gain, \( A_o \) is the effective area of the receiving antenna, \( \sigma \) is the radar cross section of the target, and \( R \) is the distance of the target to the radar. For the purposes of this study we are only interested in the effects of range and transmission power; thus, \( G, A_o, \sigma \) are considered inconsequential and the power received can be approximated by

\[ P_r \propto \frac{P_t}{R^2}. \] (2)

Given a single radar located at the origin and a single air vehicle travelling on a path from some point \( A \) to a point \( B \) some distance away, Pachter [1] showed that an objective function for minimizing \( P_r \) is

\[ J = \int_0^\ell \frac{1}{R^2(t)} \, dt, \] (3)

where \( v \) is the (constant) speed of the aircraft and \( \ell \) is the path length. A closed form solution to this problem was found using the calculus of variations, with the limitation that the aircraft traverses an angle with respect to the radar of less than 60°. Beyond this limit, the optimal path length is infinite (although the cost remains finite).

In this research, the objective cost function, equation (3), is augmented to include another radar at some arbitrary fixed-location a distance from the original radar, giving

\[ J = \int_0^\ell \left[ \frac{\alpha_1}{R_1^2(t)} + \frac{\alpha_2}{R_2^2(t)} \right] \, dt. \] (4)

The geometry of the problem is shown in Figure 1. Cartesian coordinates will be used for the formulation of flight against two threat radars as they yield a simpler performance index. Consider a transformation of the integral from the time domain to the Cartesian frame with the path defined as \( f = f(x, y(x)) = f(x, y) \). Now, \( v = dx \, dt \) or \( dt = \frac{dx}{v} \), and \( ds \), the element of arc length, is given in Cartesian coordinates by

\[ ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \sqrt{1 + y'^2} \, dx. \]

Figure 1: Two Radar Problem Geometry

Noting that the distance from each radar to some point on the path \((x, y)\) is simply,

\[ R_1(x, y) = \sqrt{(x-x_1)^2 + (y-y_1)^2}, \] (5)
\[ R_2(x, y) = \sqrt{(x-x_2)^2 + (y-y_2)^2}, \] (6)

where \((x_1, y_1)\) and \((x_2, y_2)\) are the known radar locations in Cartesian coordinates. Equation (4) can now be transformed into Cartesian coordinates, yielding the performance index

\[ J = \int_{x_0}^{x_f} \left( \frac{\alpha_1}{R_1^2(x, y)} + \frac{\alpha_2}{R_2^2(x, y)} \right) \sqrt{1 + y'^2} \, dx, \] (7)

with \( R_1(x, y) \) and \( R_2(x, y) \) defined above. The boundary conditions are the vehicle’s given initial point \((x_0, y_0)\) and final point \((x_f, y_f)\).

Gelfand [5] identified this as a special case of problems where the desire is to minimize an integral of a function with respect to its arc length, proving that the Euler equation can be represented as

\[ \ddot{y} = \frac{1}{f} \left( \frac{f'}{f} \dot{y} - f_\theta \right), \] (8)

where \( f_\theta = \frac{2\pi}{\sigma} \) and \( f_\phi = \frac{2\pi}{\phi} \). Applying this technique to equation (7) and simplifying gives the Euler equation for the two radar case. The solution of the Euler equation provides a candidate local minimum (or maximum), thus giving a locally optimal path.

2.2 Discrete Performance Index

A closed form solution to equation (7) remains elusive; thus, it is necessary to determine an approximation of the performance index given a path of discrete waypoints.

The equation of the line segment between two points \((x_j, y_j)\) and \((x_{j+1}, y_{j+1})\) and its derivative are easily determined from the point-slope form of a line, and substituting these results into equations (5)-(7) gives

\[ J_{1-2} = \int_{x_j}^{x_{j+1}} \left[ \frac{\alpha_1}{R_1^2(x_j, y_j)} + \frac{\alpha_2}{R_2^2(x_{j+1}, y_{j+1})} \right] \sqrt{1 + \left( \frac{y_{j+1} - y_j}{x_{j+1} - x_j} \right)^2} \, dx, \] (9)
with
\[ R_1(x_j, y_j) = f(x_1, y_1, x_j, y_j, x_{j+1}, y_{j+1}) \] \[ R_2(x_{j+1}, y_{j+1}) = f(x_1, y_1, x_j, y_j, x_{j+1}, y_{j+1}) \]
where \((x_1, y_1)\) and \((x_2, y_2)\) are the known locations of the radars. By substituting for the constants in equations (9)-(11), the performance index is represented as an integral of the form
\[ J_{1-2} = K \int_{s_1}^{s_2} \left( \frac{\alpha_1}{(x-a)^2 + (b x + c)^2} + \frac{\alpha_2}{(x-d)^2 + (b x + e)^2} \right) dx. \] (12)
This can be integrated through the use of tables or common symbolic mathematics software such as Mathematica, yielding a discrete approximation of the cost function, eliminating the dependence upon \(z\).

The cost can be determined for any given pair of points \((x_j, y_j)\) and \((x_{j+1}, y_{j+1})\). The total cost for a path of \(N\) line segments is simply
\[ J^* = \sum_{i=1}^{N} J_{i-1,i+1}. \] (13)
Again, this provides an accurate presentation of the continuous performance index, allowing the variational problem to be solved utilizing numerical techniques.

### 2.3 Voronoi Paths
A common graphical technique for optimal path planning against multiple radars is to make use of the Voronoi diagram. Travel along the Voronoi edge ensures that an equal amount of power is reflected to each radar. Given two equal power radars, \(a_1 = a_2\), located at \((x_1, y_1)\) and \((x_2, y_2)\), the Voronoi edge is the perpendicular bisector of the line segment connecting the radars. The comparison path for this case will be constructed of three line segments: a shortest path line from the initial point, the perpendicular bisector, and the shortest path line to the final point completing the curve. A full derivation of the intercept points can be found in [6]. For the case when the radars are of unequal transmission power, \(a_1 \neq a_2\), a weighted Voronoi diagram is used, with the resulting curve known as a circle of Apollonius. A comparison path similar to the perpendicular bisector will be constructed with the shortest path taken from the endpoints to the Voronoi edge completing the curve. Again, see reference [6] for the full technique utilized.

### 3 Results

#### 3.1 Problem Scenarios
Three parameters of the geometry were varied to examine the effects upon the optimal trajectory: the downrange distance between the radar locations \((A)\), the crossrange distance between the radar locations \((B)\), and the downrange distance between the initial and final point \((C)\). The radar and endpoint geometry for the scenarios examined is shown in Figure 2. When varying \(A\) and \(B\), the endpoints of the path were fixed at \((x_0, y_0) = (0, 0)\) and \((x_f, y_f) = (1, 0)\); when varying \(C\), the radars were fixed at \((x_1, y_1) = (0.4, 0.5)\) and \((x_2, y_2) = (0.6, -0.5)\). Two ratios of radar transmission power were examined for each of the cases: \(a_1/a_2 = 1/1\) and \(a_1/a_2 = 2/1\). The ordinary or weighted Voronoi trajectory, as applicable, was computed to be compared with the optimal trajectory.

![Figure 2: Geometry for the Two Radar Scenarios](image)

#### 3.2 Trajectory Optimization Against Two Equal Power Radars

**3.2.1 Scenario 1: Varying Downrange Radar Separation:** Intuitively, it is expected that the paths will be symmetric about the midpoint of the vector connecting the radars, and the path will bend away from the nearest radar. The results of the optimization do in fact prove this to be true. In Figure 3, the optimal path bends away from the nearest radar and intersects the downrange axis at the midpoint of the radars as the radars move towards the endpoints. It can also be seen that the path length of the optimal path is increasing as \(A\) increases. Interestingly, at the point the radars are at the same \(x\)-coordinate as the endpoints, the path length is shorter than the previous trajectory. This is because as \(A\) becomes greater than \(C\), the optimal trajectory will approach a straight line.

![Figure 3: Optimal Trajectories for Increasing A, a1 = a2](image)
line which is short compared to magnitude of $A$. As $A \rightarrow \infty$, the ratio of optimal path length to downrange distance between radars, $C/A$, and the optimal cost, $J^*$, will approach zero.

The relation of the optimal path to the Voronoi path is not evident from this scenario. It appears at first that the Voronoi path is a rough linear approximation to the optimal curve. In the limit as $A$ gets very large, however, the Voronoi edge will become perpendicular to the optimal path and the Voronoi intercepts will move to a single point, the midpoint of the Voronoi edge between the radars. The only consistently common point is the midpoint of the paths, and this is due to the symmetry of the problem.

### 3.2.2 Scenario 2: Varying Crossrange Radar Separation:
In Scenario 2, the radars are kept at a fixed downrange separation, $A$, while the crossrange, $B$, is progressively increased. Results similar to what was observed in Scenario 1 are expected; as the crossrange $B \rightarrow \infty$, the optimal path will approach a straight line, Figure 5.

For this scenario, numerical difficulties preempted finding solutions as $B \rightarrow 0$. This is likely due to the minimal cost path desiring to travel around the radars instead of between them. When problems occurred, the solution from the shooting method would travel through one of the radars. This is because the shooting method satisfies only the necessary conditions for an extremum; these cases were obviously maximizing solutions to the Euler equation.

From Figure 6 inferences can still be drawn as to the effects of varying $B$. For this formulation, as $B \rightarrow 0$, the cost, $J^*$, will approach some very large number. Since $z$ must monotonically increase, the path has nowhere to go but through the radars. In reality, the optimal trajectory would never follow this path; instead, it would travel around the radars at a much lower cost. As mentioned before, to solve this problem an alternate formulation is required.

The results of the first two scenarios followed the expectations of how the optimal trajectory would react to different radar geometries, and reinforced the fact that the path will bend away from the radars when they are close and approach the direct path as the radars move away from the endpoints. Little information could be gleaned by comparing the optimal trajectory to the Voronoi path, since it changed with each iteration.

### 3.2.3 Scenario 3: Varying Downrange Endpoint Separation:
In Scenario 3, the radar geometry is held fixed; since the radars do not move, the Voronoi edge will be constant and the only variable in the Voronoi path will be the length of the segments connecting the endpoints. The resulting optimal trajectories reveal an opportunity to exploit the Voronoi path for on-line utilization.

Figure 7 shows the optimal trajectories calculated for this scenario. As the endpoints move outward from the radars, the path extends further and further out. While these optimal paths seem excessive, the relationship between objective cost and the endpoint separation $C$, Figure 8, shows that the cost for a straight line and the optimal path are relatively close, and a straight
line path would be acceptable. Upon further analysis, the cost of the Voronoi path is seen to be nearly identical to the optimal path. Thus, the question is no longer how to get from the initial point to the final point; the question now is how to optimally approach (depart) the Voronoi edge from (to) the initial (final) point. This has crucial implications for on-line path planning. Instead of a full path optimization being performed, utilizing valuable on-line system resources, one only needs to optimize the approach and departure from the Voronoi edge.

3.3 Trajectory Optimization Against Two Unequal Power Radars

This section considers an unequal radar transmission power ratio of $\alpha_1/\alpha_2 = 2/1$. For comparison purposes, the ordinary Voronoi path is replaced by its equivalent weighted Voronoi path for the radar geometry.

3.3.1 Scenario 1: Varying Downrange Radar Separation: All of the optimal paths generated are shown in Figure 9. As expected, the optimal trajectories behaved similarly to the case when the radars were of equal power. The optimal path is now asymmetric, and bends further away from the radar in the first quadrant because it is radiating with twice the power of the other radar. The optimal trajectory bends just enough to closely follow the weighted Voronoi path for a short while and then leaves the path to meet the endpoint constraint. While not shown here, the shape of the objective cost curve is nearly the same as in Section 3.2.1, but with a slightly higher magnitude due to the radar power increase. As was the case in Section 3.2.1, one cannot easily discern the relationship between the optimal and the Voronoi path.

3.3.2 Scenario 2: Varying Crossrange Radar Separation: The trajectories generated for this case again track the same trends explained in Section 3.2.2, as shown in Figure 10. The optimal path is asymmetric, bending away from the radar of greater power. The path length of the optimal trajectories are only slightly longer than those of Section 3.2.2. To make an analogy, if the optimal paths from Section 3.2.2 were strings of constant length, the paths of this section are merely a repositioning of that string. This is likely a factor of the radar power ratio and the length difference may become more pronounced as the ratio increases. Numerical difficulties again prevented the calculation of optimal trajectories as $B$ approached the abscissa. As was the case before, the optimal trajectory would likely go around the radars and could be solved for using an alternate formulation.

3.3.3 Scenario 3: Varying Downrange Endpoint Separation: As in Section 3.2.3, by varying the parameter $C$ the best approach for travelling between the radars can be determined. The results of the optimization are shown in Figures 11-12. Whereas in Section 3.2.3 the optimal trajectory approached the perpendicular bisector, for this scenario the optimal trajectory approached the weighted Voronoi edge. Interestingly, the final portion of the path does not follow the same trend as in the equal power case. The optimal trajectory breaks off of the Voronoi edge much sooner, indicating that it is less costly to get to the final point quickly than to follow the Voronoi edge. As concluded
before, the problem now is how to optimally approach and depart from the Voronoi edge.

In contrast, varying the path endpoint separation provided a way to analyze travelling through the radars. Through direct comparison of objective cost and path length with the Voronoi diagram, it was discovered that for a feasible near-optimal trajectory, one option is to optimize the approach to and departure from the Voronoi edge. The calculation of the ordinary or weighted Voronoi edge is a simple task, and if through additional research a similarly deterministic calculation of the approach and departure paths is found, the resulting suboptimal trajectory can be quickly calculated such that it approximates the optimal solution to within acceptable limits. This is a significant result, especially for on-line trajectory generation. Partial path optimization is a much quicker and cheaper on-line calculation than performing a full path optimization, thus conserving valuable system resources for other mission tasks. Further, it may be possible to store parameterized approach/departure paths that can be splined onto the Voronoi edge for an on-line/real-time implementation.

4 Conclusion

Through the systematic variation of the downrange and crossrange radar separation, the shaping of the optimal trajectory with respect to varying radar geometries was identified. Additionally, varying the crossrange radar separation helped identify situations where different formulations or numerical methods might be necessary. By exploring the effect of these parameters, a more robust numerical optimization technique can be developed.

References


