Nonlocal Stabilization via Relay Delay Control Gain Adaptation

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Abstract

Time delay does not allow to realize an ideal sliding mode, but implies oscillations in the space of state variables. Algorithm of delayed relay control gain adaptation is suggested for nonlocal stabilization using delayed information about oscillation’s amplitudes.

1 Introduction

Time delay doesn’t allow to design the sliding mode control in the space of state variables ([5],[6]). That is why it may be singled out two main approaches to use relay control for delay systems:

- time delay compensation

The idea of delay compensation to realize the sliding mode control was formulated in [4]. Relay control algorithms for systems with delay in state and control based on delay compensation was suggested in [11],[12]. In [12] was designed the sliding mode control in the space of predictor variables for input delay systems using unit control (see the comments in [13],[9] too).

- control of oscillations amplitudes

P.I. control algorithms for oscillations control for one dimensional relay system with delay in the input have been suggested in [1].

In [5] the adaptive control algorithm for nonlocal amplitude stabilization was suggested based on extrapolation of the solutions next zero, using periodicity of scalar systems relay delay systems solution, and further adaptation of control gain. To realize the algorithm only delayed sign of solution was necessary. But algorithm from [5] was not allow to make generalization for multidimensional case, because the generally the solutions of multidimensional relay control systems are not periodic.

In [7],[8] was suggested the delayed relay control algorithm, allowing to reach local stabilization of oscillations amplitudes for controllable and weak controllable systems correspondingly.

In this paper the algorithm of delayed relay control gain adaptation is suggested using the knowledge of solutions amplitudes at delayed time moment. It allows to achieve nonlocal stabilization of oscillations amplitudes and to generalize the control algorithm for vector case.

2 Rε stabilization. Scalar case

2.1 Problem Statement

Consider the scalar control system

\[ \dot{x} = \alpha x + u(t-1), \quad 0 < \alpha < \frac{1}{2} \ln(2) \quad (1) \]

\[ x(t) = \phi(t), \quad t \in [-1, 0] \quad (2) \]

System (1) is called Rε stabilizable if for any initial condition \( \phi(t) : |\phi(0)| < R \) there exists such time delayed control law \( u(t-1) \) and the time moment \( T^* > 0 \), that \( ||x(t)||_\epsilon \) for all \( t > T^* \).

2.2 The simplest scalar control

Consider the case when \( R = \epsilon \). Then

\[ u(t) = -\alpha' \epsilon \text{sign}[x(t-1)], \quad (3) \]

where \( \alpha' \in I_\alpha \)

\[ I_\alpha = \left( \frac{\alpha e^{\alpha}}{2 - e^{\alpha}}, \frac{\alpha}{e^{\alpha} - 1} \right). \quad (4) \]

\( I_\alpha \) is not empty time interval, because \( 0 < \alpha < \frac{1}{2} \ln(2) \).
Further we will use the proposition 1 to construct the system of zero neighborhoods allowing to achieve a non-local stabilization of oscillations amplitudes in the system (1) Let us remark that for \(3^k\epsilon\) (here \(k\) is some integral number), then proposition 1 is true.

For \(R > \epsilon\) there exists \(k = N\) such that \(R \leq 3^N\epsilon\). Consider the system of neighborhoods

\[U_k = \{x : |x| < \nu_k = 3^k\epsilon\}\gamma\}, (k = 1, 2, ..., N + 1). (5)

In the paper a control law is designed guarantying that solutions of control system (1) from \(U_k\) vicinity of zero from the some moment \(T\) will stay in the \(\epsilon\) vicinity of zero for all \(t > T\).

2.3 Control Design

Suppose that:

1* \[H_k(|x|) = \begin{cases} 1, & \text{for } |x| > \nu_k \\ 0, & \text{for } |x| \leq \nu_k \end{cases} \] (6)

2* \(\alpha' \in \{\alpha, \cdot \alpha', \cdot \sqrt{\alpha}, ... \cdot \sqrt[3]{\alpha}, ..., \cdot \sqrt[3]{\alpha} \}\)

3* \[
\log_3 \left(\frac{R}{\epsilon}\right) \leq N < \log_3 \left(\frac{R}{\epsilon}\right) + 1. (7)
\]

This case the desired control law takes the form

\[u = -\alpha'\epsilon(1 + 2 \sum_{n=1}^{N} 3^{n-1} H_{\nu_n}(|x(t-1)|))\text{sign}[x(t-1)]. (8)\]

This control law has the following properties

1* The set of control values is

\[M = \{\pm \alpha', \pm 3\alpha', ..., \pm 3^N\alpha'\} \]

2* If \(\nu_k < |x(t-1)| < \nu_{k+1}\), then \(|u(t)| = 3^k\alpha'\).

3* If \(|x(t-1)| < \nu_{k+1}\), then \(|u(t)| < 3^k\alpha'\) (1 \(\leq k \leq N - 1\)).

4* \(|u(t)| \leq 3^N\alpha', (\forall t \geq 0)\)

5* \(\nu_k < 3^k\epsilon, \forall k \in \{1, 2, ..., N + 1\}\).

6* \(\nu_{k+1} - 3\epsilon < 0, \forall k \in \{0, 1, 2, ..., N\}\). In fact, taking into account (4) \(\frac{\alpha^e}{2-e^a} - \alpha' < 0\), one has

\[
\frac{2 - e^a}{2 - e^a} - \alpha' < 0.
\]

Then

\[
\frac{2 - e^a}{2 - e^a} - \alpha' = \frac{1}{\alpha}(a e^a - \alpha'(2 - e^a)) = \\
= \gamma - \frac{\alpha'}{\alpha} < 0.
\]

2.4 Control gain and vicinity radius

2.4.1 Saving of the radius:

Lemma 2 If \(\exists T \geq 0\) such that

\[|x(t)| < \nu_{k+1} \ t \in [T - 1, T]\]

and

\[|x(T)| < 3^k\epsilon\]

where \(k \in \{0, 1, ..., N - 1\}\), then

\[|x(t)| \leq \nu_{k+1}, (\forall t \geq T)\]

Proof: Let us suggest that by contradiction

\[\exists T' \geq T : |x(T')| > \nu_{k+1}\]

Then from the condition \(|x(T)| \leq 3^k\epsilon\) it follows

\[\exists T' > T : |x(t')| = 3^k\epsilon \text{ and } |x(t)| > 3^k\epsilon, (\forall t \in [t', T]),\]

and moreover

\[\exists T' > t^* : |x(T')| = \nu_{k+1} \text{ and } |x(t)| < \nu_{k+1}, (\forall t \in [t^*, T^*]).\]

Now we can suggest that \(x(T') > \nu_{k+1}\). Then \(x(t') = 3^k\epsilon\) and \(x(T') = \nu_{k+1}\).

Let us show that \(T' - t^* \geq 1\).

Let us estimate the upper bound of \(x(t)\) for \(t \in [t^*, T^*]\). Taking into account that \(|x(t)| < \nu_{k+1}\) for \(t \in [T - 1, T]\) and \(|x(t)| < \nu_{k+1}\) for \(t \in [T, T']\), one has \(|u(t)| < \alpha'3^k\epsilon\) for \(t \in [T, T^*]\). Then

\[\dot{x} \leq \alpha x + 3^k\alpha'\epsilon,

\]

and

\[x(t) \leq 3^k\epsilon((1 + \alpha')e^{\alpha(t'-t')} - \alpha').\]

For \(t = T^*\) one has

\[x(T^*) = \nu_{k+1} = 3^k\epsilon((1 + \alpha')e^{\alpha(t'-t')} - \alpha'),\]

which yields \(T' - t^* \geq 1\). Let us note that in this case from \(x(t) > 3^k\epsilon > \nu_k\) for \(t \in [t^*, T^*]\) one has \(\text{sign}[x(t -
1)] = 1 and $H_{\nu_n}(|x(t-1)|) = 1$ for $n = l, k$, $t \in [t^* + 1, T^*]$, which means

$$u(t) \geq -3^k \alpha' \varepsilon, \forall t \in [t^* + 1, T^*]. \tag{9}$$

Then

$$\dot{z}(T^*) \leq \alpha \nu_{k+1} - 3^k \alpha' \varepsilon < 0 \tag{10}$$

This means that at $t = T^*$ the function $z(t)$ is decreasing on $[T^*, T^*]$, and $z(T^*) < \nu_{k+1}$. This is a contradiction with initial suggestion.

**Corollary 1** If $|z(0)| \leq R$, then

$$|z(t)| \leq \nu_{k+1} = 3^N \varepsilon, \forall t \geq 0$$

**Proof:** It is obvious, that condition $|z(t)| \leq \nu_{k+1}$ for $t \in [T^* - 1, T]$ and $k \in \{1, \ldots, N - 1\}$ is equivalent to $|u(t)| \leq \alpha' 3^k \varepsilon \forall t \in [T, T+1]$. Lemma 1 is true even in the case when $k = N$. Then taking into account that $N \geq \log \frac{R}{\varepsilon}$, one has $|z(0)| \leq R \leq 2^N \varepsilon$.

2.4.2 Lemma about infinite numbers of zeroes:

**Lemma 3** If $|z(0)| \leq R$, then

$$\forall t \geq 0 \exists T \geq t : z(T) = 0$$

**Proof:** Let us suppose by contradiction that

$$3t^* : \forall t \geq t^* z(t) \neq 0.$$ 

Suppose that $z(t) > 0$. Then for $t > t^* + 2$ we will have $\text{sing}[z(t-1)] = 1$, then the equation (2) has the form

$$\dot{z} = \alpha x - \alpha' \varepsilon (1 + 2 \sum_{s=1}^{N} 3^{s-1} H_{\nu_s}(|z(t-1)|)).$$

In this case

$$\dot{z} \leq \alpha x - \alpha' \varepsilon \tag{11}$$

Let us show that it is not true and $3t^* \geq t^* + 1 : z(t^*) \leq \nu_1$. Then the inequality (11) yields:

$$z(t) \leq (\nu_1 - \frac{\alpha' \varepsilon}{\alpha} e^{(t-t^*)}) + \frac{\alpha' \varepsilon}{\alpha} = v(t).$$

Taking into account that the first coefficient before the exponent in the last equation is negative, one can conclude that $v(t)$ is decreasing function and $P^0 : v(0^0) = 0$, then $z(T^0) \leq 0$, which contradict with condition $z(t) > 0$. This means that $z(t) > \nu_1$. Analogously we can proof the existence of the next zero for any $k$.

2.4.3 Sufficient condition for control gain reduction:

**Lemma 4** If

$$|z(t)| \leq \nu_{k+1}, \forall t \geq T, \tag{12}$$

then there exists such $T_1$, that

$$|z(t)| \leq \nu_k, \forall t \geq T_1.$$

**Proof:** 1. Consider the case:

$$\alpha' \geq \frac{1}{3 \varepsilon^a - 1}.$$ 

Then from condition (12) and (50) it follows that

$$\dot{z} \leq \alpha x + \alpha' \varepsilon 3^k, \forall t \geq T + 1.$$ 

Suppose that $t = t^*$ is zero of $z(t)$ and $t^* > T + 1$. Then

$$z(t) \leq \frac{\alpha' \varepsilon 3^k}{\alpha} e^{\alpha(t-t^*)} = \frac{\alpha' \varepsilon 3^k}{\alpha}. \tag{13}$$

For $t = t^* + 1$ we will have

$$z(t^* + 1) \leq \frac{3^k \alpha' \varepsilon}{\alpha} e^{3^k \alpha' \varepsilon} - \frac{3^k \alpha' \varepsilon}{\alpha} = \frac{3^k \alpha' \varepsilon}{\alpha}.$$ 

Then from the last inequality

$$z(t^* + 1) \leq \frac{3^k \alpha' \varepsilon}{\alpha} e^{3^k \alpha' \varepsilon} - \frac{3^k \alpha' \varepsilon}{\alpha} = 3^k \varepsilon.$$ 

For $t \in [t^*, t^* + 1]$ one has $|z(t)| \leq 3^{k-1} \varepsilon \leq \nu_k$ and $z(t^* + 1) \leq 3^{k-1} \varepsilon$. From lemma 1 it follows that $z(t) \leq \nu_k \forall t \geq t^*$. Analogously we can have $z(t) \geq -\nu_k$.

2. Consider the case

$$\alpha' > \frac{1}{3 \varepsilon^a - 1}.$$ 

Let $t = t^*$ is the zero of the solution $z(t)$ such that

$$t^* \leq T + 2 - \frac{1}{\alpha} \ln(1 + \frac{\alpha}{3 \varepsilon^a}). \tag{13}$$

Then for $t \geq t^*$ one has

$$z(t) \leq \frac{\alpha' \varepsilon 3^k}{\alpha} e^{\alpha(t-t^*)} - \frac{\alpha' \varepsilon 3^k}{\alpha} = v(t).$$

Consequently the function $v(t)$ is increasing for some $t^1 \geq t^* : v(t^1) = 3^k \varepsilon$. Then from the last inequality

$$\frac{\alpha' \varepsilon 3^k}{\alpha} e^{\alpha(t-t^*)} - \frac{\alpha' \varepsilon 3^k}{\alpha} = 3^k \varepsilon,$$

and

$$t^1 - t^* = \frac{1}{\alpha} \ln(1 + \frac{\alpha}{3 \varepsilon^a}) \tag{14}$$
Consider two cases

α) \(|z(t)| \leq \nu_k \) for \( t \in [T+1, t^*] \), then taking into account (13), (14), one has \( t^* - T \geq 1 \), which means that for \( t \in [T+1, t^*] \), \( \|z(t)\| \leq 2 \), \( \|z(t^*)\| \leq 3^{k-1} \epsilon \), and one can conclude that lemma 3 follows from lemma 1.

β) \( \exists t \in [T, t^*] : |z(t)| > \nu_k \). Then from the continuity of \( z(t) \) it follows \( \exists t^2 > T : z(t^2) = \nu_k \) and \( z(t) \leq \nu_k \), \( \forall t \in [t^2, t^*] \).

Consequently, \( |z(t)| \leq \nu_k \), \( \forall t \in [t^2, t^*] \). Let us estimate the lower band of \( z(t) \) for \( t \in [t^2, t^*] \). Then

\[
\dot{z} \geq \alpha z - \alpha' \epsilon z^3
\]

\( \forall k \),

and

\[
z(t) \geq (\nu_k - \alpha' \epsilon z^3) e^{\alpha(t-t^2)} + \alpha' \epsilon z^3
\]

Let us rewrite this inequality at \( t = t^* \) in the form:

\[
0 \geq (\nu_k - \alpha' \epsilon z^3) e^{\alpha(t^*-t^2)} + \alpha' \epsilon z^3
\]

Then

\[
0 \geq (\epsilon z^3 - (\alpha + \alpha') \epsilon - \alpha') (e^{\alpha(t^*-t^2)} - 1) \geq 3 \alpha' \epsilon z^3
\]

and

\[
0 \geq ((\alpha + \alpha') \epsilon - 4 \alpha') e^{\alpha(t^*-t^2)} + 3 \alpha'
\]

\( (2 \alpha' + (2 - \epsilon) \alpha') - \alpha \epsilon \geq 3 \alpha' \epsilon \).

Now from (4) it follows that \( 2 - \epsilon \alpha \geq 0 \) and

\[
t^* - t^1 \geq \frac{1}{\alpha} \ln \frac{3 \alpha'}{(4 - \epsilon \alpha) \alpha' - \alpha e^\alpha}
\]

Taking into account the last inequality, one has

\[
t^* - t^2 = t^1 - t^* + t^* - t^2 \geq \frac{1}{\alpha} \ln (1 + \frac{\alpha}{3 \alpha'}) + \frac{1}{\alpha} \ln \frac{3 \alpha'}{(4 - \epsilon \alpha) \alpha' - \alpha e^\alpha} = \frac{1}{\alpha} \ln \frac{3 \alpha' + \alpha}{(4 - \epsilon \alpha) \alpha' - \alpha e^\alpha}
\]

Let us show that

\[
\frac{1}{\alpha} \ln \frac{3 \alpha' + \alpha}{(4 - \epsilon \alpha) \alpha' - \alpha e^\alpha} \geq 1.
\]

Then

\[
\frac{3 \alpha' + \alpha}{(4 - \epsilon \alpha) \alpha' - \alpha e^\alpha} \geq \frac{e^\alpha}{e^\alpha}
\]

and consequently \( t^* - t^2 \geq 1 \).

2.5 Main Theorem

**Theorem 2** For any initial condition \( \phi(t) : |\phi(0)| \leq R \), there exists such time moment \( t = T \), that for any \( t > T \) \( |z(t)| < \epsilon \).

**Proof:**

1. Let us show that there exists such time moment \( t = T_1 \), that

\[
|z(t)| < \nu_1, \forall t \geq T_1.
\]

Following the corollary 1 one has \( |z(t)| \leq \nu_{N+1} \) for \( t \geq 0 \). Lemma 3 yields that there exists such time moment that \( t = t^1 \) \( |z(t)| \leq \nu_N, \forall t \geq t^1 \). Analogously for \( N \)th step one has

\[
|z(t)| \leq \nu_k \forall t \geq t^n.
\]

2. Inequality (15) yields that for \( t \geq t^n + 1 \) one has \( |u(t)| \leq \alpha' \epsilon \). Moreover, from lemma 2 it follows that \( \exists T \geq t^n + 1 : z(T) = 0 \). Let us show that \(|z(T)| \leq \epsilon, \forall t \geq T \). Suppose by contradiction that, if \( \exists T \geq T : z(T) > \epsilon \), then \( \exists T^* > T : z(T^*) = 0 \). Let us find the upper band of \( z(t) \) for \( t \in [T^*, \tilde{T}] \). Then

\[
\dot{z} \leq \alpha z + \alpha' \epsilon \cdot z(T^*) = 0.
\]

Then

\[
z(t) \leq \frac{\alpha' \epsilon}{\alpha} e^{\alpha (t - T^*)} - \frac{\alpha' \epsilon}{\alpha}.
\]

and for the time moment \( T^* + 1 \) the last inequality takes the form

\[
z(T^* + 1) \leq \frac{\alpha' \epsilon}{\alpha} e^{\alpha} - \frac{\alpha' \epsilon}{\alpha} = \frac{\alpha' \epsilon - 1}{\alpha} \epsilon \leq \epsilon.
\]

There is switching of control at the time moment \( T^* + 1 \) and

\[
\dot{z} = \alpha z - \alpha' \epsilon,
\]

but \( z(T^* + 1) \leq \epsilon \leq \nu_1 \). Consequently from 6° it follows

\[
\dot{z}(T^* + 1) \leq \alpha \nu_1 - \alpha' \epsilon < 0.
\]

This means that for \( t \geq T^* + 1 \) the solution \( z(t) \) will decrease until the next switching moment. This means that at some time moment \( z(T^*) = 0 \). This equality contradicts with condition \( z(t) > 0, \forall t \in (T^*, \tilde{T}) \).

3. \( \epsilon \) Stabilization for Vector Case

\[
\frac{dx}{dt} = Ax + Bu
\]

where \( z \in \mathbb{R}^n \) is vector of state space, \( u \in \mathbb{R}^n \) is control vector, \( A, B \) are real matrix. Consider the set of pairs \( Q \) of smooth mappings \( S, F \)

\[
S : \mathbb{R}^n \rightarrow \mathbb{R}^k, F : \mathbb{R}^k \rightarrow \mathbb{R}^m, S = (S_1, S_2, ..., S_k)^T.
\]
Let us find the delayed relay feedbacks in form $u = F(u_{re}[S_1(x(t-1)), \nu^1, N_1], ..., u_{re}[S_k(x(t-1)), \nu^k, N_k])$, where $N_k \in \mathbb{N} \cup \mathbb{R}$, $\nu^i = (\nu_1^i, \nu_2^i, ..., \nu_N^i)$,

$$u_{re}[\rho, \nu^i, N_i] = (1 + 2 \sum_{n=1}^{N_i} 3^{n-1} H_{v_i}(n^3) \text{sign}[^\rho]),$$

and $(S, F) \in Q$.

Let us denote as $x(t)$ the solution to the system (16) with initial conditions $x(t) = p(t), -1 \leq t \leq 0$.

**Definition 5** The zero solution to the system (16) is said to be $Re$ stabilizable, if for any $R > \varepsilon > 0$ there are integer $k > 0$, the pair $(S, F) \in Q$, the set of vectors $\{\nu^1, \nu^2, ..., \nu^k\}$, and time moment $T > 0$, such that from the inequality $\sup_{t \in [T, \infty]} ||x(t)|| < R$ it follows that $\sup_{t \in [T, \infty]} ||x(t)|| < \varepsilon$.

### 4 $Re$ Stabilization of Single Input Systems

Consider the single input system in form

$$\dot{z} = Ax + bu,$$  \hspace{1cm} (17)

where $b = (b_1, b_2, ..., b_n)^T$, $u$ is scalar control. Let us denote by $\varphi_A = \lambda^n + \alpha_1 \lambda^{n-1} + ... + \alpha_n$ the characteristic polynomial for matrix $A$, the pair $(A, b)$ is controllable, and, consequently, the vectors $\{b, Ab, ..., A^{n-1}b\}$ are linearly independent.

Let us introduce the new basis into $\mathbb{R}^n$ in form

$$e_1 = A^{n-1}b + \alpha_1 A^{n-2}b + ... + \alpha_{n-1}b,$$

$$e_2 = A^{n-2}b + \alpha_1 A^{n-3}b + ... + \alpha_{n-2}b,$$

$$...$$

$$e_{n-1} = Ab + \alpha_1 b,$$

$$e_n = b.$$

System (17) in this basis takes the form

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = y_3,$$

$$\dot{y}_n = -\alpha_n y_1 - \alpha_{n-1} y_2 - ... - \alpha_2 y_{n-1} + u,$$

and it is possible to rewrite the systems (17) and (18) in the form of $n - th$ order equation

$$y_1^{(n)} + \alpha_1 y_1^{(n-1)} + ... + \alpha_n y_1 = u.$$  \hspace{1cm} (19)

Suppose that matrix $A$ satisfies the condition (i) the characteristic equation of matrix $A : \varphi_A(\lambda) = 0$ has only one positive root $\lambda_1 \in (0, \frac{1}{\ln 2})$, and the other roots of this equation have negative real parts.

In such case the polynomial $\varphi_A(\lambda)$ takes the form

$$\varphi_A(\lambda) = (\lambda - \lambda_1) \psi(\lambda),$$

where the polynomial $\psi(\lambda) = \lambda^{n-1} + \beta_{n-1} \lambda^{n-2} + ... + \beta_1$ has only the roots with negative real parts. Then equation (19) takes the form

$$\frac{d}{dt} \dot{z} = \lambda_1 \dot{z},$$

Suppose that $z(t) = \frac{d}{dt} \psi(\lambda) = y_1^{(n-1)}y_1 + ... + \beta_1 y_1(t), \dot{y}_1(t)$. For $z(t)$ we will have scalar equation

$$\frac{d}{dt} \dot{z} = \lambda_1 \dot{z} + u,$$

From previous section it follows that the zero solution of that equation $\dot{z} = \lambda_1$ is $\varepsilon$ stabilizable with control

$$u = -\alpha' u_{re}[z(t-1), \nu, N].$$

Returning to the (19), we can rewrite the control as

$$u = -\alpha' u_{re}[y_1^{(n-1)}y_1(t-1) + ... + \beta_1 y_1(t-1), \nu, N].$$

For the variables $y_1, y_2, ..., y_n$ one has

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = y_3,$$

$$...$$

$$\dot{y}_n = -\alpha_n y_1 - \alpha_{n-1} y_2 - ... - \alpha_2 y_{n-1} + \alpha_1 y_n + u,$$

The final form of control system for the variables $x = P^{-1} y$ is the following

$$\dot{z} = Ax - \alpha' \beta u_{re}[\gamma, z(t-1), \nu, N],$$

where $\gamma = P^{-1} \beta$, $\beta = (\beta_1, \beta_2, ..., \beta_{n-1}, 1)$.

**Theorem 3** System (17) under condition (i) is $Re$ stabilizable.

### Conclusion

Algorithm of delayed relay control gain adaptation is suggested for nonlocal stabilization using delayed information about oscillation’s amplitudes.

### References


