H∞ Sub-optimization in Dynamic Back-stepping Multiple Surface Control

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Abstract

This paper considers the optimization problem for dynamic multiple surface control of nonlinear systems in strict feedback form with additive uncertainties. Backstepping combined with multiple surfaces sliding mode control is the control design method. Integral filters are used to estimate the derivative of the composite reference state at each step to avoid explosion of the number of terms. The problem can be described as H∞ sub-optimization in s-space where s is the coordinates determined by sliding functions s = (s1,...,sn). Here the sub-optimization is with respect to s as a whole instead of each state respectively. These sliding gains can be determined by solving a set of triangularly coupled algebraic inequalities, which is easy to implement. This is a partial optimization in the sense that optimization is not with respect to integral filter gains. A pertinent third order example is given.

1 Introduction

Back-stepping multiple surfaces sliding mode control combined with integral filters for control design of nonlinear systems in a strict feedback form with unmatched uncertainties has been proposed by previous work in [22]. This method combines three features in nonlinear control design:

(1) Sliding mode method: In ideal sliding mode, dynamics of the closed-loop system are restricted to a sliding manifold, essentially a lower dimensional sub-manifold, on which a stable steady state will be reached in finite time or asymptotically [4,18,21,23]. This approach is generally recognized to be robust to matched uncertainties [17,19].

(2) Back-stepping in control design logic: In each backward step, a proper sliding surface is chosen. Thus multiple sliding surfaces naturally result in the end. This is a promising way in dealing with additive unmatched uncertainties in nonlinear models [14,22].

(3) Integral filters: Integral filters are used to estimate the derivatives of composite reference state (signal) at each step. Thus analytic differentiation is avoided. Dynamic feedback naturally results. Using integral filters has several advantages in both theory and real-time implementation. In theory, the differentiability of the reference state can be removed. Condition required for the existence and uniqueness of solution is Lebesgue integrability [6,12]. Thus, switching or saturation function can be used in sliding reachability condition.

This paper considers H∞ partial sub-optimization problem for dynamic multiple surface control design. The sub-optimization here is partial in the sense that it does not include the optimization of the filter dynamics.

Previous work [7] also considered optimization in control design for nonlinear systems (without uncertainty) in strict feedback form using back-stepping design logic which can be interpreted as recursive Lyapunov design. It adopts an optimization approach focusing on softening the controller. [14,15] considered the control design for such a type of system with parameter uncertainties. Work in this paper is different in the following points:

(a) Unmatched additive uncertainties are explicitly addressed;
(b) Sliding mode control and back-stepping design logic are combined to deal with unmatched uncertainties;
(c) Integral filters are used to calculate the derivative of composite reference state (measured signals) at each design step instead of using analytic differentiation to avoid term number explosion;
(d) It is shown that, by proper parameterization, the optimization problem can be formulated as a nonlinear H∞ type. [7] tries to directly find optimal controller.

Nonlinear H∞ method has been well developed theoretically in recent years [1,2,3,5,8,9,10,11,16,20,24]. Its theoretical foundation can either be differential game theory or passivity theory [9,25]. They both result in solving a nonlinear Hamiltonian-Jacobian-Isaac (HJI) differential inequality for state feedback or two coupled HJI inequalities for measurement feedback. Usually, they are nonlinear partial differential inequalities which are difficult to solve analytically. This hinders its practical application in nonlinear control design. However, it has a good application for systems with saturation nonlinearity in the control channel [26]. For linear systems, this is reduced to solving an algebraic Riccati Inequality [9].

Eventually, H∞ sub-optimization techniques are used to choose sliding gains and the storage function. Due to the special structure of the strict feedback form, the solution of the HJI inequality here can be simplified as solving consecutively a set of triangularly coupled second order algebraic inequalities.

This paper is structured as follows. Section 2 introduces preliminary work necessary for this paper including main results in sliding mode control and nonlinear
Section 3 reformulates the dynamic multiple surface sliding mode control. Section 4 formulates the optimization problem and then solve it in a $H_{\infty}$ sub-optimization approach. Section 5 gives a pertinent example of dimension 3. Some concluding remarks are made in Section 6.

2 Background

Consider the following SISO nonlinear system in strict feedback form with unmatched uncertainties

$$ \dot{x}_i = f_i(x_1, x_2, ..., x_n) + \Delta_i(x_1, x_2, ..., x_n) $$

where

$$ |\Delta_i| \leq \rho_i \| x(0) \| + l_i $$

$$ x(0) = (x_1, ..., x_i) $$

$$ \rho_i \geq 0, l_i \geq 0 $$

Here the assumption is that

1. $\phi_i(i = 1, ..., n)$ are Lebesgue integrable with respect to its entries and that in the region of interests;
2. $\rho_i, l_i (i = 1, ..., n)$ are known constants.

Control task:

(a) to make $x_i$ to asymptotically track a continuous trajectory $x_{id}(t)$ (not necessarily differentiable);
(b) to render the closed loop system robustly stable with respect to the uncertainties.

3 Dynamic Back-stepping Multiple Surface Control

This method combines the advantages of different control design methods. The main features of this design method can be described as:

1. Sliding mode method: Using proper sliding surfaces and sliding reachability conditions;
2. Back-stepping is used in control design logic in formulating error dynamics. In each step, a proper sliding surface is chosen. Thus multiple sliding surfaces naturally result in the end. This is a promising way in dealing with additive unmatched uncertainties in nonlinear models [22];
3. Integral filters are used to calculate the derivatives of reference signals at each consecutive step. Thus analytic differentiation of reference signal is avoided.

The formulation of the problem is slightly different from that in [22].

3.1 Control Design Method

Design a controller with the following procedure is suitable for any sliding reachability condition. This method was first proposed in [22].

Step 1 Let

$$ S_1 = x_1 - x_{id} $$

Formally differentiate $S_1$ and set

$$ \dot{S}_1 = -\gamma_1(S_1) + S_2 + \Delta_1 $$

$$ \dot{S}_2 = x_2 - x_{id} $$

where $x_{id}$ is to be determined. Besides, $x_{id}$ may not be differentiable. To avoid this difficulty, use the constructed derivative $x_{id} \cdot \dot{x}$ from an integral filter to replace the formal derivative $\Delta_1(x_{id})$ in $\dot{S}$.

$$ \dot{x}_1(t) = \frac{1}{\tau_1} (-z_1(t) + x_{id}(t)) $$

and use the following estimation

$$ x_{id}(t) \approx x_{id} \cdot \dot{x} = \frac{1}{\tau_1} (-z_1(t) + x_{id}(t)) $$

It is obtained that

$$ \dot{x}_1 = x_1 \cdot \dot{x} - \gamma_1(S_1) + S_2 + \Delta_1(x_1) $$

which leads to

$$ x_{2d} = x_{id} \cdot \dot{x} - \gamma_1(S_1) - \phi_1(x_1). $$

Thus, to make $x_1$ to track $x_{id}$, it is necessary and sufficient that $x_1$ tracks $x_{2d}$ and $x_2$ tracks $x_{id}$.

Step $i (i = 1, 2, ..., n - 2)$ Let

$$ S_{i+1} = x_{i+1} - x_{id} $$

where $x_{id}$ is to be determined. Formally differentiate $S_{i}$ and set

$$ \dot{S}_i = -\gamma_i(S_i) + S_{i+1} + \Delta_i $$

Then use the filtered signal $x_{id} \cdot \dot{x}$ to replace $\frac{\Delta_i}{\Delta_i}$ in $\dot{S}_i$ as follows:

$$ \dot{z}_i = \frac{1}{\tau_i} (-z_i + x_{id}) $$

$$ \dot{x}_{id} \approx x_{id} \cdot \dot{x} = \frac{1}{\tau_i} (-z_i + x_{id}) $$
It is obtained that
\[
\dot{x}_i = \dot{x}_{id} + \varphi_i(x_1, x_2, ..., x_i) + \Delta_i
\]
which leads to
\[
x_{i+1} = \dot{x}_{id} + \varphi_i(x_1, x_2, ..., x_i)
\]

Step n-1 Formally differentiate
\[
S_n = x_n - x_{nd}
\]
\[
x_{nd} = x_{n-1} + \dot{x}_{nd} - \varphi_{n-1}(x_1, x_2, ..., x_n)
\]
Set
\[
\dot{S}_n = -\gamma_n(S_n) + \Delta_n
\]
and replace \( \dot{x}_{nd} \) with the filtered signal \( \dot{x}_{nd} \) constructed from a filter:
\[
\dot{x}_n = \frac{1}{\tau_n} (-z_n + z_{nd})
\]
\[
\dot{x}_{nd} = \frac{1}{\tau_n} (-z_n + z_{nd})
\]
It is obtained that
\[
\dot{x}_n = \dot{x}_{nd} + \gamma_n(S_n) + \Delta_n
\]
\[
\varphi_n(x_1, x_2, ..., x_n) + u + \Delta_i
\]
from which the controller is constructed as
\[
u = [x_{nd} - \gamma_n(S_n) - \varphi_n(x_1, x_2, ..., x_n)]
\]

3.2 Closed-loop system

In (s-2)-space, the closed-loop system is composed of
\[
\dot{\xi}_1 = -\gamma_1(s_1) + s_2 + \Delta_1
\]
\[
\dot{\xi}_i = -\gamma_i(s_i) + s_{i+1} + \Delta_i
\]
\[
\dot{\xi}_{i-1} = -\gamma_{i-1}(s_{n-1}) + s_n + \Delta_{i-1}
\]
\[
\dot{\xi}_n = -\gamma_n(s_n) + \Delta_n
\]
and
\[
\dot{\xi}_1(t) = \frac{1}{\tau_1} (-z_1(t) + z_{id}(t))
\]
\[
\dot{\xi}_2(t) = \frac{1}{\tau_2} (-z_2(t) + z_{id}(t))
\]
\[
\dot{\xi}_i(t) = \frac{1}{\tau_i} (-z_i(t) + z_{id}(t))
\]
\[
\dot{\xi}_n(t) = \frac{1}{\tau_n} (-z_n(t) + z_{id}(t))
\]
Semi-global stability of the closed-loop system was achieved in [22].
or equivalently, to find \( K \) and \( K_0 \) such that the system (4.1) is passive with respect to the storage function

\[
V(s) = s^TPs
\]

where \( P \) is a diagonal positive definite matrix to be determined. If this is achieved, the uncertainty attenuation with internal stability is then guaranteed. Furthermore, the solution is optimal in the sense that the performance index

\[
J = \|z(t)\|^2
\]

is sub-optimized. The choice of the function \( z(t) \) is flexible. For simplicity, the following penalty function is used:

\[
z = \alpha \|\| + (1 - \alpha) \|v\|
\]

\( 0 < \alpha \leq 1 \) \hspace{1cm} (4.2)

**Remark 4.1** Two points are emphasized as follows:

(1) As long as \( \alpha > 0 \), \((A,K)\) is a controllable pair.

(2) For convenience, it is assumed that in \( s \)-coordinate, the uncertainty has a similar bound as those in strict feedback form:

\[
|\Delta_i| \leq \rho_i \|\| + l_i, \hspace{0.5cm} i = 1, ..., n \hspace{1cm} (4.3)
\]

where \( l_i, \rho_i \geq 0 \) are known. From which, one obtains that

\[
|\Delta| \leq \rho \|\| + l
\]

\[
\rho = \sqrt{\rho_1^2 + \cdots + \rho_n^2}
\]

\[
l = \sqrt{l_1^2 + \cdots + l_n^2}
\]

It is noted that the uncertainty type (2.2) in \( s \)-coordinate can always be transferred to the uncertainty type (4.3) if one properly restrict the region of interests according to the back-stepping design procedure.

The main result of this paper is stated as follows.

**Theorem 4.1** Consider system (4.1) with uncertainty bound (4.3).

(1) Choose the attenuation rate \( \sigma \) and weight \( \alpha \) in penalty function (4.2) such that

\[
0 < \sigma < 1
\]

\[
1 \leq \alpha \leq 1
\]

(2) Choose a diagonal matrix \( P = \text{diag}[p_1, ..., p_n] \) to satisfy

\[
p_i > \frac{2(1-\alpha)^2 + 2(1-\alpha)^4 \sqrt{(1-\alpha)^2 + 2(1-\alpha)^2 \rho_i^2}}{4(1-\alpha)^3}
\]

\[
2 \leq i \leq n - 1
\]

(3) Choose the sliding gain matrix \( K = \text{diag}[k_1, ..., k_n], k_i > 0, \)

\[
k_i < \frac{p_i^2 + \sqrt{P_i^2 - 4(\alpha - 1)^2 P_i^2 + 2(1-\alpha)^2 + (1-\alpha)^2 P_i^2}}{4(1-\alpha)^3}
\]

\[
2 \leq i \leq n - 1
\]

(4) Choose \( K_0 = \text{diag}[k_0_1, ..., k_0_n] \) to satisfy

\[
k_{0i} > \max \left\{ l_i, \frac{\alpha}{2(1-\alpha)} \right\}
\]

Then

(1) the uncertainties in the \( s \)-dynamics are attenuated

\[
\int_{t_0}^{t} \|z(\theta)\|^2 d\theta \leq \sigma^2 \int_{t_0}^{t} \|u(\theta)\|^2 d\theta
\]

for all \( t \geq t_0 \) with respect to the storage function \( V = s^TPs \).

(2) \( z = \alpha \|\| + (1 - \alpha) \|v\| \)

\( u = Ks \)

and \( w(t) = \rho \|\| + l \) is the upper bound for uncertainty;

Proof is omitted due to length restriction.

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**5 Example**

Using integral filters has several advantages in real-time implementation. The right hand side of the system is only required to be Lebesgue (practically) integrable.

Consider the following uncertain system in a strict feedback form with uncertainty:

\[
\dot{x}_1 = x_2 + x_3^2 + \Delta_1
\]

\[
\dot{x}_2 = x_3 + \sin x_1 + x_2 x_2 + \Delta_2
\]

\[
\dot{x}_3 = x_1^3 + x_2 x_2 + u + \Delta_3
\]

\( \Delta_1(t) = 0.1 (0.5 - \text{rand}(1)) x_1 \sin x_1 \)

\( \Delta_2(t) = 0.15 (0.5 - \text{rand}(1)) x_2 \cos x_1 \)

\( \Delta_3(t) = 0.25 (0.5 - \text{rand}(1)) x_2^2 \)

where \( \text{rand}(1) \) is the random function in Matlab. The control task is to make \( x_1 \rightarrow x_{1d} = \sin(t) \) (asymptotical tracking)

If restrict the region of interests in

\[
|x_1| \leq 1
\]

\[
|x_2| \leq 1
\]

then

\[
|\Delta| \leq \rho_i \|x_i(0)\|, \hspace{0.5cm} i = 1, 2, 3
\]

\( \rho_1 = 0.1, \rho_2 = 0.15, \rho_3 = 0.25 \)

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$x_{id}$ denotes the reference state for $x_i$, $i = 1, 2, 3$.

Step 1 Let
\[ x_{id} = \sin(t) \]
\[ x_{id}.dot = \cos(t) \]
where the first order filter is not necessary.

Step 2
\[ x_{2d} = x_{id}.dot - k_1 s_1 - x_1^2 \]
\[ s_2 = x_2 - x_{2d} \]
\[ x_{id}.dot = \cos(t) \]
where the derivative of $x_{2d}$ is to be replaced with
\[ \dot{x}_1 = \frac{1}{\tau_1} (-z_1 + x_{2d}) \]
\[ x_{id}.dot = \frac{1}{\tau_1} (-z_1 + x_{2d}) \]

It is obtained from the sliding reachability condition that
\[ x_{3d} = (-z_1 + x_{2d}.dot) / \tau_1 - k_2 s_2 - (\sin x_1 + x_1 x_2) \]

Step 3 Let
\[ s_3 = x_3 - x_{3d} \]
In the derivative of $x_{3d}$, the derivative of $x_{3d}$ is replaced by $x_{3d}.dot$ which is the result from a first order filter
\[ \dot{s}_3 = \frac{1}{\tau_2} (-z_2 + x_{3d}) \]
\[ x_{3d}.dot = \frac{1}{\tau_2} (-z_2 + x_{3d}) \]

From the sliding reachability condition, the control is solved out as
\[ u = \frac{1}{\tau_2} (-z_2 + x_{3d}) - k_3 s_3 - x_1^2 - x_2 x_3 \]

The parameters are chosen as follows.
\[ \sigma = 0.426 \]
\[ \tau_2 = \tau_3 = 0.02 \]

The initial condition is chosen as $x_0 = [0.1; -0.15; 0.2; -0.05; 0.01]$.

In the simulation, the gains are calculated according to the rules provided in last section. $\alpha = 0.88, 0.98$ are used for different simulations, from which one can see its effect on performances. As discussed before, difference $\alpha$ corresponds to different performance index. As expected from the theory, $\alpha = 0.98$ (Fig. 2) should corresponds to better tracking than $\alpha = 0.88$ (Fig. 1). For different choice of $\alpha$, different choices of $p_1$ and $k_1$ are listed below. The choices of $p_1$ and $k_1$ abide by according to the principle that $p_1$ tends to be their lower bound value while $k_1$ tend to be their upper bound value.

\[ \alpha = 0.88 \left\{ \begin{array}{ll} p_1 = 0.3806 & p_2 = 0.4469 & p_3 = 0.4081 \\ k_1 = 3.8585 & k_2 = 3.8551 & k_3 = 3.6447 \end{array} \right. \]
\[ \alpha = 0.98 \left\{ \begin{array}{ll} p_1 = 0.0964 & p_2 = 0.1279 & p_3 = 0.2298 \\ k_1 = 11.6758 & k_2 = 9.0204 & k_3 = 4.7526 \end{array} \right. \]

From the simulation results, although $\alpha = 0.98$ leads to a good tracking, there is a prominent transient overshoot for control $u$. This is undesirable in practice. This may be due to the use of high gain first order integral filter. This will be considered further in future work.

6 Concluding Remarks

The optimization problem for the dynamic backstepping multiple surface sliding mode control for nonlinear systems in a strict feedback form with additive matched and unmatched uncertainties has been considered. A first order integral filter is used to estimate
the derivative of the composite reference state in each
design step, which effectively avoids analytical differ-
entiation and thus term number explosion. This is a
partial sub-optimization problem with respect to the s-
dynamics, which turns out to be $H_\infty$ sub-optimal gain
choice. It leads to a systematic rule as simple as solv-
ing a set of triangularly coupled second order algebraic
equations and thus suitable for real-time implementa-
tion.

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