Relationships between Input-Output Stability and Exponentially Stable Periodic Orbits*

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Abstract

In this paper, we present new results concerning the relationships between the input-output and Lyapunov stability of nonlinear system possessing a periodic orbit. Both Lyapunov theory and input-output theory are well-developed for nonlinear system in [6, 8, 9, 10, 14, 16]. In Lyapunov stability it has been studied how Lyapunov stability can be used to establish $L_2$ stability of nonlinear systems around equilibrium point and the interrelationships between the two theories was presented [15]. We can expect the similar results on the stability of periodic orbits. Although there are many results on the stability of periodic orbit, most of them are concerning the stability of a periodic orbit in the state-space. On the other hand, the notion of input-output stability of periodic orbit is very useful. This is because we are often not able to get explicit solution of periodic orbit so that it is difficult to estimate the behaviour of orbit in the presence of external disturbances.

We have seen that many stability criteria regarding equilibrium point are easier to state and prove in the input-output stability than in the Lyapunov stability [6, 8, 9, 10, 14, 16]. However, many problems regarding periodic orbit stability are more naturally stated in the Lyapunov stability than in the input-output stability. In this context, if one could conclude input-output stability from Lyapunov stability, it would be possible to use the techniques and results from the theory to solve the problems of the other. Thus, we study the relationships between the input-output and Lyapunov stability of periodic orbit.

This paper is organized as follows. We introduce local coordinates, transverse dynamics, and Converse Theorem in Section 2. In Section 3, $L_2$ stability around periodic orbit is defined and relationships between input-output and Lyapunov stability of periodic orbit is presented. In Section 4, $L_2$ gain analysis is introduced and an example is shown to illustrate $L_2$ gain analysis. Then conclusion follows.

1 Introduction

In this paper, we present new results concerning the relationships between the input-output and Lyapunov stability of nonlinear system possessing a periodic orbit. Both Lyapunov theory and input-output theory are well-developed for nonlinear system in [6, 8, 9, 10, 14, 16]. In Lyapunov stability it has been studied how Lyapunov stability can be used to establish $L_2$ stability of nonlinear systems around equilibrium point and the interrelationships between the two theories was presented [15]. We can expect the similar results on the stability of periodic orbits. Although there are many results on the stability of periodic orbit, most of them are concerning the stability of a periodic orbit in the state-space. On the other hand, the notion of input-output stability of periodic orbit is very useful. This is because we are often not able to get explicit solution of periodic orbit so that it is difficult to estimate the behaviour of orbit in the presence of external disturbances.

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This paper is organized as follows. We introduce local coordinates, transverse dynamics, and Converse Theorem in Section 2. In Section 3, $L_2$ stability around periodic orbit is defined and relationships between input-output and Lyapunov stability of periodic orbit is presented. In Section 4, $L_2$ gain analysis is introduced and an example is shown to illustrate $L_2$ gain analysis. Then conclusion follows.

2 Preliminaries

The set of all n-dimensional vectors $x = (x_1,\cdots,x_n)^T$, where $x_1,\cdots,x_n$ are real numbers, defines the n-dimensional Euclidean space defined by $\mathbb{R}^n$. We shall consider the class of the p-norms, defined by

$$||x||_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p},$$

1 $\leq p < \infty$ and

$$||x||_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The space $L^p_{\mathbb{R}^n}$ for $1 \leq p < \infty$ is defined as the set of all piecewise continuous functions $x: [0,\infty) \to \mathbb{R}^n$ such that

$$||x||_{L^p_{\mathbb{R}^n}} = \left( \int_0^\infty ||x(t)||^p dt \right)^{1/p} < \infty$$

and the space $L^\infty_{\mathbb{R}^n}$ is defined as the set of all piecewise continuous, uniformly bounded functions with the norm

$$||x||_{L^\infty_{\mathbb{R}^n}} = \sup_{t \in [0,\infty)} ||x(t)|| < \infty.$$

Consider the smooth dynamical system

$$\dot{x} = f(x)$$

on $\mathbb{R}^n$ and suppose that $\eta \subset \mathbb{R}^n$ is a periodic orbit of (1) with period $T$.

Using the distance function

$$d(x,\eta) := \min_{y \in \eta} ||x - y||$$
an $\epsilon$-neighborhood of $\eta$ can be specified as

\[ B_\epsilon(\eta) := \{ x \in \mathbb{R}^n : d(x, \eta) < \epsilon \}. \]

The orbit $\eta$ is stable if trajectories starting near $\eta$ stay near $\eta$, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in B_\delta(\eta) \to \phi(t)(x) \in B_\epsilon(\eta)$ for all $t \geq 0$ ($\phi(t)$ is the flow of the vector field $f$); $\eta$ is asymptotically stable if it is stable and trajectories starting near $\eta$ converge to $\eta$, i.e., there is a $\delta > 0$ such that $d(\phi(t)(x), \eta) \to 0$ as $t \to \infty$ for all $x \in B_\delta(\eta)$; $\eta$ is exponentially stable if it is asymptotically stable and the convergence is exponential, i.e., there exist $\delta, M, \lambda > 0$ such that $d(\phi(t)(x), \eta) \leq d(x, \eta) M e^{-\lambda t}$ for all $x \in B_\delta(\eta)$. This type of stability is often called orbital stability to distinguish it from the often used concept of Lyapunov stability [5].

It is clear that the behavior that we are interested in is determined by the $n-1$ dimensional transverse dynamics. To highlight these dynamics, it is shown in [7] we can construct a set of local coordinates $(\theta, \rho) \in S^1 \times \mathbb{R}^{n-1}$ as follows.

First, parameterizing $S^1$ by $\theta \in [0, T]$ (i.e., the closed segment $[0, T]$ with the two endpoints identified), define a mapping $\theta \mapsto y(\theta)$ by solving

\[ \frac{dy(\theta)}{d\theta} = f(y(\theta)) \]

with $y(0) \in \eta$ arbitrary (but fixed). Then, the $\theta$ coordinate of a point $x$ in a neighborhood of $\eta$ is defined by

\[ \theta = \psi_1(x) := \arg \min_{y \in \{y(\theta) : \theta \in [0, T] \}} \|y(\theta) - x\|^2. \]

Note that the function $\psi_1(\cdot)$ is a smooth since $f(\cdot)$ is and the minimizer is unique on a small neighborhood of $\eta$. Next, letting $\psi_i(\cdot)$, $i = 2, \ldots, n$, be a set of functions that are independent and vanish on $\eta$, the $\rho$ coordinate of $x$ is given by

\[ \rho_{i-1} = \psi_i(x), \quad i = 2, \ldots, n. \]

The following proposition shows that it is always possible to find such functions.

**Proposition 2.1** Suppose that $\eta$ is a periodic orbit of $\dot{x} = f(x)$. Then, there exist $n-1$ independent functions $\psi_i(\cdot)$, $i = 2, \ldots, n$, defined on a neighborhood of $\eta$ that vanish on $\eta$.

Then we know that $(\theta, \rho)$ is a well defined set of coordinates in a neighborhood of $\eta$.

**Proposition 2.2** The mapping $x \mapsto (\theta, \rho)$ given by $(\theta, \rho) = \Psi(x)$ in (2) and (3) is a diffeomorphism on a neighborhood of $\eta$.

Roughly speaking, since the $\rho$ coordinates are transverse to the periodic orbit at each point of $\eta$, the stability of the orbit is largely determined by the behavior of the $\rho$ coordinates. This is seen more clearly by examining the system dynamics in the $(\theta, \rho)$ coordinates.

**Proposition 2.3** The dynamics of the nonlinear system (1) in a neighborhood of the periodic orbit $\eta$ have the form

\[ \begin{align*}
\dot{\theta} &= 1 + f_1(\theta, \rho) \\
\dot{\rho} &= A(\theta)\rho + f_2(\theta, \rho)
\end{align*} \]

where $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ satisfy

\[ f_1(\theta, 0) = 0, \quad f_2(\theta, 0) = 0, \quad \text{and} \quad \frac{\partial f_2(\theta, 0)}{\partial \rho} = 0. \]

The canonical form (4) provides a useful characterization of a periodic orbit.

**Proposition 2.4** The orbit $\eta$ is an exponentially stable orbit of (1) if and only if the transverse linearization

\[ \frac{d\rho}{d\theta} = A(\theta)\rho \]

is asymptotically stable.

The linearization of a nonlinear system about an exponentially stable equilibrium point is commonly used in the construction of quadratic Lyapunov functions for the nonlinear system. Such functions are constructed by solving a Lyapunov equation. Roughly speaking, the transverse linearization takes the place of the equilibrium point linearization and a periodic Lyapunov equation takes the place of the Lyapunov equation.

Associated with the periodic linear system (5) is the periodic Lyapunov equation

\[ P'(\theta) + A(\theta)^T P(\theta) + P(\theta)A(\theta) + Q(\theta) = 0 \]

where

\[ P'(\theta) = \frac{dP}{d\theta}(\theta) \]

and $Q(\theta)$ is a continuous, periodic matrix. The following result is well known (see, e.g., [12], [1]).

**Theorem 2.5** Suppose that (5) is asymptotically stable and that $Q(\theta)$ is positive definite for all $\theta$. Then (6) has a unique periodic solution $P(\theta)$ that is positive definite for all $\theta$.

We are now ready to construct quadratic Lyapunov functions that prove the exponential stability of periodic orbit $\eta$.

**Theorem 2.6** (Converse Theorem) [7] Suppose that $\eta$ is an exponentially stable periodic orbit of (1). The Lyapunov function

\[ V(x) = \rho^T P(\theta) \rho \]

proves the exponential stability of $\eta$. That is, on a neighborhood of $\eta$, $V(\cdot)$ satisfies

\[ k_1\|x\|_\rho^2 \leq V(x) \leq k_2\|x\|_\rho^2 \]

for some $k_1, k_2 > 0$. The canonical form (4) provides a useful characterization of a periodic orbit.
and
\[ \frac{\partial V}{\partial x} \leq k_1 \|x\|_n \]
and
\[ V(x) \leq -k_3 \|x\|^2_0 \]
for some positive constants \( k_1, k_2, k_3, k_4 \) and
\[ \|x\|_n := \inf_{y \in \eta} \|x - y\|. \]

See [7, 21] for the proofs of propositions and theorems in this section.

3 Main Results

Consider the nonlinear system

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^l \), \( f \) and the columns of \( g \) are smooth vector fields, and the elements of \( h \) are smooth functions. We may think of \( x = (x_1, \ldots, x_n) \) as local coordinates for a smooth state space manifold of dimension \( n \).

Let \( \eta \) be a periodic orbit of (10) for \( u = 0 \) and suppose that \( h \) vanishes on \( \eta \). We consider the system (10) as a system whose input-output relationship is represented by

\[ y = Hu \]

where \( H \) is a mapping that specifies \( y \) in terms of \( u \). With an extended space \( L^m \) defined by

\[ L^m = \{ u | u_r \in L^m, \forall \tau \geq 0 \} \]

where \( u_r \) is a truncation of \( u \) defined by

\[ u_r(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq \tau \\ 0, & \text{if } t > \tau \end{cases} \]

the mapping \( H \) is defined as a mapping from an extended space \( L^m \) to an extended space \( L^l \). A mapping \( H : L^m \rightarrow L^l \) is said to be causal if the value of the output \( (Hu)(t) \) at any time \( t \) depends only on the values of the inputs up to time \( t \).

This is equivalent to

\[ (Hu)_r = (Hu_r)_r. \]

With the space of input and output signals defined as above, we can define input-output stability.

**Definition 3.1** A mapping \( H : L^m \rightarrow L^l \) is finite-gain \( L \) stable (around \( \eta \)) if there exist nonnegative constants \( \gamma \) and \( \beta \) such that

\[ \| (Hu)_r \|_L \leq \gamma \|u_r\|_L + \beta \]

for all \( u \in L^m \) and \( \tau \in [0, \infty) \).

The system is small-signal finite-gain \( L \) stable (around \( \eta \)) if it is true for a certain class of signal spaces \( L \), i.e. small signal \( u \in L_{\infty} \).

**Definition 3.2** A mapping \( H : L^m \rightarrow L^l \) is small-signal finite-gain \( L \) stable (around \( \eta \)) if there exists a positive constant \( \delta < \infty \) such that

\[ \|(Hu)_r\|_L \leq \gamma \|u_r\|_L + \beta \]

is satisfied for nonnegative constant \( \gamma \) and \( \beta \), and for all \( u \in L^m \cap L_{\infty} \) and \( \tau \in [0, \infty) \) with \( \sup_{t \in [0, \infty)} \|u(t)\| \leq \delta \).

If the nonlinear system (10) which has an exponentially stable closed orbit is disturbed, the ultimate boundedness of the solution of the disturbed system can be shown in the same way as the ultimate boundedness of solution about an equilibrium point. First, we construct a converse Lyapunov function for the exponentially stable periodic orbit. Then, we analyze its stability with respect to disturbance.

**Proposition 3.1** Consider a nonlinear system of the form

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^l \), \( f \) and the columns of \( g \) are smooth vector fields, and the elements of \( h \) are smooth functions. Suppose \( \eta \) as an exponentially stable periodic orbit of the unforced system of (11) such that

\[ \dot{x} = f(x) \]

and \( h(x) \) vanishes on \( \eta \). Furthermore, suppose that there exists \( k_\eta \) such that

\[ \|g(x)\| \leq k_\eta \quad \forall \eta \in U. \]

and, since \( h(x) \) vanishes on \( \eta \) and is smooth, there exists \( k_\eta \) such that

\[ \|y\| := \|h(z)\| \leq k_\eta \|x\|_n. \]

Then the system (11) is small-signal finite-gain \( L \) stable around \( \eta \) for all \( \tau \in [0, \infty) \).

**Proof:** By the converse theorem, there exists a \( C^1 \) Lyapunov function \( V : U \rightarrow \mathbb{R}_+ \) and positive constants \( k_1, k_2, k_3, k_4 \) such that

\[ k_1 \|x\|^2 \leq V(x) \leq k_2 \|x\|^2_0 \]

\[ \frac{\partial V}{\partial x} \leq k_3 \|x\|_n \]

\[ V(\eta)(x) \leq -k_4 \|x\|^2_0 \]

for \( x \in B_\eta(\eta) \subset U \) where \( V(\eta)(x) \) denotes the derivative of \( V \) along the trajectory of (12).

If we evaluate the derivative of \( V \) along the trajectory of (11), we get

\[ V(11)(x) = \frac{\partial V}{\partial x}(f(x) + g(x)u) \]

\[ \leq -k_4 \|x\|^2 + k_3 k_2 \|x\|_n \|u\| \]

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for \( |x| < r \). Let \( W = V^\frac{1}{2} \). Then \( W \) is continuously differentiable except at \( x \in \eta \). Hence if \( x \notin \eta \), we have \( 2W \dot{W} = V \) so that (17) becomes

\[
2W \dot{W} \leq -k_1\|x\|^2 + k_3k_5\|x\|\|u\|
\leq \frac{k_1}{k_2}W^2 + \frac{k_3k_5}{\sqrt{k_1}}\|u\|.
\]

(18)

Dividing both sides of (18) by \( 2W \) gives

\[
\dot{W} \leq -\frac{k_4}{2k_2}W + \frac{k_3k_5}{2\sqrt{k_1}}\|u\|.
\]

Now using comparison principle [lo] gives

\[
W(t) \leq e^{-t\frac{k_4}{2k_2}}W(0) + \frac{k_3k_5}{2\sqrt{k_1}} \int_0^t e^{-(t-\tau)\frac{k_4}{2k_2}}\|u(\tau)\|d\tau
\]

or

\[
\|x\|^2 \leq \sqrt{\frac{k_3}{k_1}}\|x(0)\|e^{-\frac{k_4}{2k_2}t} + \frac{k_3k_5}{2\sqrt{k_1}} \int_0^t e^{-(t-\tau)\frac{k_4}{2k_2}}\|u(\tau)\|d\tau.
\]

It can be easily verified that

\[
\|x(0)\| < r \sqrt{\frac{k_1}{k_2}} and \sup_{0 \leq \tau \leq s} \|u(\sigma)\| < \frac{r_k k_4}{k_2 k_3 k_5}
\]

ensures that \( \|x(t)\| < r \). Hence \( x(t) \) stays within \( B_r(\eta) \).

Using (13), we have

\[
\|y\| \leq c_1e^{-at} + c_2 \int_0^t e^{-a(t-\tau)}\|u(\tau)\|d\tau
\]

(19)

where

\[
c_1 = \sqrt{\frac{k_3}{k_1}}\|x(0)\|,\ c_2 = \frac{k_3k_5}{2\sqrt{k_1}},\ a = \frac{k_4}{2k_2}.
\]

Therefore, for \( u \in L_\infty^p \) and \( \|u\|\infty \leq \frac{r_k k_4}{k_2 k_3 k_5} \) for all \( p \in [1, \infty) \), it can be easily verified that

\[
\|y\|\infty \leq \gamma\|u\|\infty + \beta
\]

where

\[
\gamma = \frac{c_2}{a},\ \beta = c_1\delta
\]

\[
\delta = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{1}{\tau_p}\right)^{1/p}, & \text{if } p \in [1, \infty). \end{cases}
\]

\[\Box\]

Remark We could get the same result with the relaxed conditions on \( f(x) \), \( g(x) \) and \( h(x) \) such that \( f(x) \) is continuously differentiable, \( g(x) \) is bounded on a neighborhood \( U \) of \( \eta \), \( h(x) \) vanishes on the periodic orbit \( \eta \) and satisfies Lipschitz condition (13) around periodic orbit \( \eta \).

Corollary 3.2 In Proposition 3.1, if \( u(t) \in L_\infty^\infty \) and \( u(t) \to 0 \) as \( t \to \infty \), then \( y(t) \to 0 \) as \( t \to \infty \).

Proof: It is shown that if \( u(t) \in L_\infty^\infty \) and \( u(t) \to 0 \) as \( t \to \infty \), so does the convolution on the right side of (19) [4].

\[\Box\]

4 \( L_2 \) Gain Analysis

\( L_2 \) stability plays an important role in system analysis since its square-integrable signals can be viewed as finite-energy signals. Nonlinear system (20) can be represented as an input-output map from a disturbance \( u \) to the output \( y \). We are often interested in finding not only the input-output stability but also the \( L_2 \) gain of the system.

Given a nonlinear system possessing a periodic orbit, it is shown that we can construct a quadratic Lyapunov function according to the Converse Theorem [7]. Furthermore, it is studied that we can compute the induced \( L_2 \) gain for the nonlinear system [3]. In Section 3, we showed that given a nonlinear system which has an exponentially stable periodic orbit, the system is small-signal finite-gain \( L_2 \) stable around the periodic orbit for all \( p \in [1, \infty) \).

However, there is nonlinear system which does not have an exponentially stable periodic orbit but has \( L_2 \) gain around periodic orbit. In this section we will illustrate the argument with example.

Consider the nonlinear system

\[
\dot{x} = f(x) + g(x)u
\]

\[
y = h(x)
\]

(20)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^p \), \( f \) and the columns of \( g \) are smooth vector fields, and the elements of \( h \) are smooth functions. Let \( \eta \) be a periodic orbit of (20) for \( u = 0 \) and suppose that \( h \) vanishes on \( \eta \). A function \( V \) is positive definite on \( \eta \) if \( V(x) = 0 \), \( x \in \eta \), and \( V(x) > 0 \) for \( x \in B_r(\eta) \backslash \eta \) for some open neighborhood \( B_r(\eta) \) of \( \eta \).

The nonlinear system (20) has \( L_2 \) gain (around \( \eta \)) less than \( \gamma \) if there is a constant \( \delta > 0 \) and \( \epsilon > 0 \) such that, for all \( x_0 \in B_r(\eta) \), for all \( u \in L_2 \) with \( \|u(t)\| \leq \delta \) \( \forall t \in [0, \infty) \),

\[
\int_0^\infty \|y(\tau)\|^2 d\tau < \gamma^2 \int_0^\infty \|u(\tau)\|^2 d\tau + 2V(x_0).
\]

(21)

Sufficient conditions for showing that the \( L_2 \) gain (around \( \eta \)) of a system is less than \( \gamma \) look quite similar to the conditions for \( L_2 \) gain analysis around an equilibrium point [13].

Theorem 4.1 \[8\] Consider the nonlinear system (20) and suppose that \( V \in C^1 \) is positive definite on \( \eta \). We have the following implications:

(A) \( \Leftrightarrow \) (B) \( \Rightarrow \) (C)

(A) There is an open neighborhood \( B_r(\eta) \) of \( \eta \) such that

\[
\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)g^T(x)\frac{\partial^2 V}{\partial x^2}(x) + \frac{1}{2}h^T(x)h(x) < 0
\]

(22)

for \( x \in B_r(\eta) \) \backslash \eta .

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for $x \in B_\epsilon(\eta) \setminus \eta$ and $u \in \mathbb{R}^m$.

(C) The system has $L_2$ gain less than $\gamma$ around $\eta$.

Inequality (22) is known as the Hamilton-Jacobi inequality. We showed that, given an exponentially stable periodic orbit, a continuously differentiable $V(x)$ can be constructed using the local coordinates presented in Section 2. However, unlike Proposition 3.1, in Theorem 4.1 we do not require the periodic orbit of the unforced system to be exponentially stable. This point is illustrated by the following example.

Example Consider the planar system

$$\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}$$

where

$$f(x) = \begin{bmatrix} -(\sqrt{x_1^2 + x_2^2} - 1)^2 - \sqrt{x_1^2 + x_2^2} - x_2 \\
-(\sqrt{x_1^2 + x_2^2} - 1)^3 - \sqrt{x_1^2 + x_2^2} + x_1 \end{bmatrix},$$

g(x) = \begin{bmatrix} 0 \\
\sqrt{x_1^2 + x_2^2} - 1 \end{bmatrix},
$$
$$h(x) = \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2.$$

For the example, we can take $V(x) = \frac{1}{2}(\sqrt{x_1^2 + x_2^2} - 1)^2$ as Lyapunov function to show that the system has an asymptotically stable periodic orbit. In addition, taking the Lyapunov function for the solution of the Hamiltonian-Jacobi inequality (22), we get the following Hamiltonian-Jacobi inequality

$$\begin{align*}
\partial V(x)f(x) + \frac{1}{2} \partial^2 V(x)g(x)g^T(x) &\partial^2 V(x) + \frac{1}{2} h^T(x)h(x)

&= -\frac{1}{2} \left(\sqrt{x_1^2 + x_2^2} - 1\right)^4 + \frac{1}{2} \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2 \\
&= -\frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right) \left(\sqrt{x_1^2 + x_2^2} - 1\right)^4 < 0
\end{align*}$$

for all $x \in B_\epsilon(\eta) \setminus \eta$ and $\gamma > 1$.

Thus we could conclude that this system is small-signal finite-gain $L_2$ stable and its $L_2$ gain is less than $\gamma$ ($\gamma > 1$). Note that the condition of Proposition 3.1 are not satisfied by this example since the periodic orbit of the unforced system is not exponentially stable. This example shows that we could still calculate $L_2$ gain using the Hamiltonian-Jacobi Inequality (22), although the system does not have exponentially stable periodic orbit.

5 Conclusion

In this paper, we presented new results concerning the relationships between the input-output and Lyapunov stability of nonlinear system possessing a periodic orbit. Definition of small-signal finite-gain $L_\infty$ stability around periodic orbit was introduced. Furthermore, we showed $L_\infty$ stability of exponentially stable periodic orbit using quadratic Lyapunov functions for the periodic orbit. Finally, the $L_2$ gain analysis was presented with Hamiltonian-Jacobi inequality along with an example.

References