Biconical Antennas with Unequal Cone Angles

Surendra N. Samaddar, Life Senior Member, IEEE, and Eric L. Mokole, Member, IEEE

Abstract—The problem of radiation and reception of electromagnetic waves associated with a spherically capped biconical antenna having unequal cone angles $\psi_1$ and $\psi_2$ is investigated. Both cones that comprise a bicone are excited symmetrically at the apices by a voltage source so that the only higher order modes are TM. A variational expression for the terminal admittance is derived. Under the wide-angle approximation, expressions for the radiated field, the effective height, and the terminal admittance are obtained. In addition, limiting values of these quantities are derived for electrically small and electrically large wide-angle bicones. The results for arbitrary cone angles are new and subsume results that appear in the existing literature as special cases such as where $\psi_1 = \psi_2$ or $\psi_2 = \pi/2$. Moreover, the approximations of this paper are more accurate than many in the literature. It is argued that the radiation pattern of an electrically small cone is proportional to $\sin \theta$, which is similar to that of a short dipole; whereas the pattern behaves like $1/\sin \theta$ for electrically large cones. The parameter $\theta$ is the angle from the bicone's axis of symmetry to the observation direction. Consequently, the direction of maximum radiation changes with exciting frequency for a bicone of fixed length. Although most of the analyses are presented in the frequency-domain, time-domain responses of bicones are discussed for some special cases that are similar to situations considered by Harrison and Williams. In particular, the time-domain radiated field and the received voltage are shown to depend on the input's passband and on the match between the source and the bicone.

I. INTRODUCTION

In this paper, the biconical antenna analyses provided by Schelkunoff [1], Smith [2], Tai [3]–[5], Papas and King [6], [7], and Sandler and King [8] are generalized by considering axially symmetric bicones having unequal cone angles. The geometry of the antenna configuration is shown in Fig. 1, and $(r, \theta, \phi)$ are spherical coordinates. The common axis of the two cones is oriented along the $z$ axis and the cone angles $\psi_2$ and $\psi_1$ satisfy $0 < \psi_1 < \pi/2$ and $0 < \psi_2 \leq \pi - \psi_1$. Relative to $\theta$, the lateral surfaces of the upper and lower cones correspond to $\theta = \psi_1$ and $\theta = \pi - \psi_2$, respectively. By a proper choice of $\psi_1$ and $\psi_2$, the exit aperture of the antenna can be adjusted so that the radiated power will be directed in a desired direction, which is one of the motivations behind this investigation. Furthermore, knowledge of biconical antenna characteristics is helpful in understanding why TEM horns, V-antennas, triangular plates, and bow-tie antennas are very wide band.

The cones are excited symmetrically at the apices so that only TEM and TM modes are generated. The field components can then be expressed in terms of a scalar function $\Pi(r, \theta)$ [1]–[5], which is equivalent to the radial component of the vector potential and is azimuthally invariant. Formal expressions for the field components are presented first in each of the regions $0 \leq r \leq a$ and $r \geq a$, where $a$ is the length of each cone and corresponds to the radius of the sphere in Fig. 1. On using these field components, a variational expression for the terminal admittance is derived in Sections II-A and II-B. When $\psi_1 = \psi_2$, this variational expression agrees with Tai's result [5].

Also of interest are the wide-band and ultrawide-band behaviors of biconical antennas, which have application in surveillance and communications. Papas and King [6] demonstrated that both the input resistance and reactance of wide-angle bicones having cone angles exceeding 40° are very slowly varying functions of frequency for very wide frequency ranges. Furthermore, they showed that the higher-order TM modes in the antenna region $(0 \leq r \leq a)$ can be neglected for wide-angle bicones. Under this wide-angle approximation, general results for a bicone’s effective length, input impedance, terminal admittance, and radiated field are derived (Section II-C). These results are further analyzed for the limiting cases of electrically small and large wide-angle cones. Radiation patterns for two specific wide-angle bicones with $a = 20$...
Inches are discussed to contrast the behavior of equal cone angles versus that of unequal cone angles: 1) $\psi_1 = \psi_2 = 53.1^\circ$ and 2) $\psi_1 = 53.1^\circ$ and $\psi_2 = 70^\circ$. In Section III, the transient responses of wide-angle biconical antennas for some special cases, similar to the situations considered by Harrison and Williams [9], are studied.

II. ANALYSIS

In this section, formal expressions of the electric- and magnetic-field components for the antenna region ($0 \leq r \leq a$) and its complement ($r > a$) are expanded in terms of series involving Bessel ($J_\nu$), Legendre ($P_\nu$), and Hankel ($H_\nu^{(2)}$) functions. The unknown coefficients of these expansions and the terminal admittance $Y_t$ of the bicone at $r = a$ are determined. The expression for $Y_t$ is recast in a variational form, which is evaluated to obtain a series representation of $Y_t$. At this point, the expressions for the fields, the input impedance $Z_m$, the effective height $h_e$, and $Y_t$ are simplified by using a wide-angle approximation for the cones. These results are reduced further by applying approximations for two special cases: electrically small ($ka \ll 1$) and electrically large ($ka \gg 1$) wide-angle cones.

Within the antenna region ($0 < r < a$ and $\psi_1 < \theta < \pi - \psi_2$), the components of the electric and magnetic fields are given by

$$\hat{\omega}c_0 r^2 E_r = -\sum_{\nu_p} \frac{a_{\nu_p}}{2\pi} S_{\nu_p}(kr) T_{\nu_p}(\theta)$$

$$r E_\theta = \frac{I_0 Z_0}{2\pi \sin\theta} \left[ (1 + Z_c Y_t) e^{-ik(r-a)} + (1 - Z_c Y_t) e^{ik(r-a)} \right] - iZ_0 \sum_{\nu_p} \frac{a_{\nu_p}}{2\pi \nu_p (\nu_p + 1) S_{\nu_p}(ka)} \frac{\partial}{\partial \theta} T_{\nu_p}(\theta)$$

$$r H_\phi = \frac{I_0}{2\pi \sin\theta} \left[ (1 + Z_c Y_t) e^{-ik(r-a)} - (1 - Z_c Y_t) e^{ik(r-a)} \right] - \sum_{\nu_p} \frac{a_{\nu_p}}{2\pi \nu_p (\nu_p + 1) S_{\nu_p}(ka)} \frac{\partial}{\partial \theta} T_{\nu_p}(\theta)$$

where $S_{\nu_p}'$ denotes differentiation of $S_{\nu_p}$ with respect to its argument and $\{a_{\nu_p}\}$ and $I_0$ are constants that must be determined. Furthermore, by [1], $Z_0 = \sqrt{\mu_0/\epsilon_0} = 120\pi$

$$Z_c(\psi_1, \psi_2) = \frac{Z_0}{2\pi} \int_\psi^\pi \frac{1}{\sin\theta} d\theta = \frac{Z_0}{2\pi} \ln[\cot(\psi_1/2) \cot(\psi_2/2)]$$

$$T_{\nu_p}(\theta) = P_{\nu_p}(\cos\theta) P_{\nu_p}(-\cos\psi_1) - P_{\nu_p}(-\cos\theta) P_{\nu_p}(\cos\psi_1)$$

$$S_{\nu_p}(kr) = \sqrt{kr} J_{\nu_p + \frac{1}{2}}(kr)$$

Equation (2b) satisfies the boundary condition that $E_r$ vanishes on the surface of the bicone for $0 < r < a; T_{\nu_p}(\psi_1) = 0$ and $T_{\nu_p}(\pi - \psi_2) = 0$. In general, the index $\nu_p$ runs over a countable number of noninteger values, which are determined by solving the transcendental equation $T_{\nu_p}(\pi - \psi_2) = 0$ for each $p = 1, 2, \cdots$. Moreover, taking the limit as $r$ approaches $a$ in (1b) yields

$$I_0(\psi_1, \psi_2) = \frac{a}{2Zc(\psi_1, \psi_2)} \int_\psi^{\pi-\psi_2} E\phi(\alpha, \theta) d\theta.$$  

Outside the antenna region ($r > a$)

$$i\omega c_0 r^2 E_r = -\sum_{\nu = 1}^{\infty} \frac{b_{\nu}}{2\pi R_{\nu}(ka)} R_{\nu}(kr) F_{\nu}(\cos\theta)$$

$$r E_\theta = -iZ_0 \sum_{\nu = 1}^{\infty} \frac{b_{\nu}}{2\pi l(l+1) R_{\nu}(ka)} \frac{\partial}{\partial \theta} P_{\nu}(\cos\theta)$$

$$r H_\phi = -\sum_{\nu = 1}^{\infty} \frac{b_{\nu}}{2\pi l(l+1) R_{\nu}(ka)} \frac{\partial}{\partial \theta} P_{\nu}(\cos\theta)$$

where $R_{\nu}$ is the derivative of $R_{\nu}$ with respect to its argument, $\{b_{\nu}\}$ are unknown constants, and

$$R_{\nu}(kr) = \sqrt{kr} H_{\nu+\frac{1}{2}}(kr).$$

A. Determination of $a_{\nu_p}$, $b_{\nu}$, and $Y_t$

Representations for the unknown coefficients $a_{\nu_p}$ and $b_{\nu}$ and the terminal admittance $Y_t$ of (1) and (4) are determined by applying the continuity conditions at $r = a$ and the orthogonality relations for $T_{\nu_p}$ and $P_{\nu}$. In particular, since $E_r$ and $r H_\phi$ are continuous at $r = a$ for $\psi_1 < \theta < \pi - \psi_2$

$$\sum_{\nu_p} b_{\nu} T_{\nu_p}(\theta) = a_{\nu_p} T_{\nu_p}(\theta), \quad \psi_1 < \theta < \pi - \psi_2$$

where $\mu_1 = \cos\psi_1$, $\mu_2 = \cos\psi_2$

$$\hat{b}_\nu = \frac{b_{\nu}\pi}{Z_0 I_0}$$

and

$$g_l(\mu_1, \mu_2) = P_l(\cos\psi_1) - (-1)^l P_l(\cos\psi_2).$$

When $\psi_1 = \psi_2 = \psi$, (7a) reduces to

$$Y_t = \left( \frac{Z_0}{2\pi Z_c} \right)^2 \sum_{l=1}^{\infty} \frac{2\hat{b}_l}{l} P_l(\cos\psi)$$

which is (14) of [3].
Setting (1b) and (4b) equal at $r = a$ leads to

$$\frac{iz_0}{2\pi a} \sum_{l=0}^{\infty} \frac{b_l M_l}{l(l+1)} \frac{\partial}{\partial \theta} P_l(\cos \theta)$$

$$= \begin{cases} \varepsilon_0(\varepsilon), & \psi_1 \leq \theta \leq \pi - \psi_2 \\ 0, & 0 \leq \theta \leq \psi_1 \text{ and } \pi - \psi_2 \leq \theta \leq \pi \end{cases}$$

(8a)

where

$$E_\theta(\theta) = \frac{iz_0 Z_0}{\pi a \sin \theta} - \frac{iz_0}{2\pi a} \sum_{l=0}^{\infty} \frac{a_{2l} N_{2l}}{l(l+1)} \frac{\partial}{\partial \theta} I_{l+1}(\theta)$$

(8b)

$$M_l = \frac{R_l^2(ka)}{R_0(ka)} \text{ and } N_{2l} = \frac{S_{2l}(ka)}{S_{2l}(ka)}.$$  

(8c)

Multiplying (8a) by $\sin \theta \frac{\partial}{\partial \theta} P_r(\cos \theta)$ for positive integer $r$, integrating from $\theta = 0$ to $\theta = \pi$, and noting that $E_\theta$ vanishes on the spherical caps of the metallic bicone yield

$$g_r(\mu_1, \mu_2) = -\frac{iz_0}{2\pi a} \sum_{l=0}^{\infty} \frac{a_{2l} N_{2l}}{l(l+1)} \frac{\partial}{\partial \theta} I_{l+1}(\theta)$$

$$+ \frac{iz_0 b_l M_l}{\pi(2r+1) \Omega(\psi_2)}$$

(9)

where $\mu = \cos \theta$ (see (10a) at the bottom of the page) and

$$\Omega(\psi_2) = 1 + \delta_{\psi_2, \frac{\pi}{2}} = \begin{cases} 1, & \psi_2 \neq \frac{\pi}{2} \\ 2, & \psi_2 = \frac{\pi}{2} \end{cases}.$$  

(10a)

The function $\Omega(\psi_2)$ is introduced to account for the situation when the cone with angle $\psi_2$ is replaced with an infinite perfect conductor at the plane $z = 0$ (Fig. 2) that occurs in [6]–[9]. After applying the orthogonality of $T_{l+1}(\theta)$ on $\psi_1 \leq \theta \leq \pi - \psi_2$ to (5), one obtains

$$a_{2l} = \sum_{l=1}^{\infty} b_l I_{l+1, 2l}.$$  

(11a)

The quantity $I_{l+1, 2l}$ is given by

$$I_{l+1, 2l} = \int_{-\mu_2}^{\mu_1} [T_{l+1}(\mu_2)]^2 d\mu$$

$$= \frac{1}{2l+1} \left[ (1 - \mu_2^2) \frac{\partial}{\partial \mu} T_{l+1}(\mu_1) \frac{\partial}{\partial \mu} T_{l+1}(\mu_2) - (1 - \mu_2^2) \frac{\partial}{\partial \mu} T_{l+1}(\mu_2) \frac{\partial}{\partial \mu} T_{l+1}(\mu_1) \right].$$

(11b)

Next, substituting (7b) for $\hat{b}_r$ and (11a) for $a_{2l}$ in (9) results in

$$g_r(\mu_1, \mu_2) = \frac{\hat{b}_r M_r}{\Omega(\psi_2) I_0(2r+1)}$$

$$- \frac{i\pi(r+1)}{2Z_0} \sum_{l=0}^{\infty} \sum_{l=1}^{\infty} \frac{b_l b_{2l} N_{2l}}{l(l+1)} I_{l+1, 2l} I_{l+1, 2l} I_{l+1, 2l} I_{l+1, 2l}$$

(12)

which involves only the unknowns $\{b_l\}$. Finally, replacement of $g_r(\psi_1, \psi_2)$ in (7a) by the right side of (12) provides an expression for the terminal admittance $Y_{l}$

$$Y_l = \frac{iz_0}{4\pi I_0 Z_0^2} \sum_{r=1}^{\infty} \frac{l_r^2 M_r}{\Omega(\psi_2) r(r+1)(2r+1)}$$

$$- \sum_{l=1}^{\infty} \sum_{l=1}^{\infty} \sum_{l=1}^{\infty} \frac{b_l b_{2l} N_{2l}}{l(l+1)} I_{l+1, 2l} I_{l+1, 2l} I_{l+1, 2l} I_{l+1, 2l}$$

(13)

in terms of $\{b_l\}$ only. Hence, (11a), (12), and (13) represent the coupling among $a_{2l}$ and $b_l$ more simply than (5) and (6).

As mentioned in the introduction, when TM waves or complementary waves in the antenna region are negligible $a_{2l}$ can be neglected, which is accurate for wide-angle cones according to [2] and [6]. Letting $a_{2l} = 0$ in (9) implies the following approximate value of $\hat{b}_r$:

$$\hat{b}_r \simeq -\frac{i\pi(2r+1)}{2Z_0 M_r} g_r(\mu_1, \mu_2) \Omega(\psi_2).$$

(14)

Subject to the wide-angle cone condition ($a_{2l} \simeq 0$), which may be called the zero-order approximation, the expression for the terminal admittance becomes

$$Y_{l0} = -\frac{iz_0}{4\pi Z_0^2(\psi_1, \psi_2)} \sum_{r=1}^{\infty} \frac{(2r+1)\Omega(\psi_2)}{r(r+1)M_r} g_r(\mu_1, \mu_2).$$

(15)

$$I_{l+1, 2l} = \int_{-\mu_2}^{\mu_1} T_{l+1}(\mu_1) P_r(\mu_2) d\mu = \frac{(1 - \mu_2^2) P_r(\mu_1) \frac{\partial}{\partial \mu} T_{l+1}(\mu_1) - (-1)^r(1 - \mu_2^2) P_r(\mu_2) \frac{\partial}{\partial \mu} T_{l+1}(\mu_2)}{r(r+1) - l^2(l+1)}$$

(10a)
Equation (15) can also be obtained from (13) by neglecting the term consisting of the triple sum and then replacing \( \delta \) with \((7b)\) and \((14)\).

**B. Evaluation of \( Y_1 \)**

The representation of \( Y_1 \) is first recast in terms of the unknown aperture field \( E_{a} \), which is then expanded in a series involving the sequence \( \{T_{\nu_p}\} \). Finally, \( Y_1 \) is derived by requiring that \( E_{a} \) give a stationary value of \( Y_1 \).

With the aid of \( (8a) \) and \( (8b) \), express \( b_{p} \) and \( a_{\nu_p} \), as follows in terms of the aperture field \( E_{a}(\theta) \) at \( \theta = \alpha \):

\[
b_{p} = \frac{2\pi \alpha \Omega(\psi_{2})}{Z_{0}M_{r}I_{p}\nu_{p}} \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} T_{p}(\cos\theta) d\theta
\]

\[
a_{\nu_p} = \frac{2\pi \alpha}{Z_{0} N_{\nu_p} I_{p} \nu_{p}} \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} T_{\nu_p}(\theta) d\theta
\]

\[
I_{r\nu_p} = \int_{\psi_{1}}^{\psi_{2}} [P_{r}(\cos\theta)]^2 \sin\theta d\theta = \frac{2}{2s+1}.
\]

Substituting \( (3), (16a), \) and \( (16b) \) for \( I_{r\nu_p}, b_{p}, \) and \( a_{\nu_p} \), respectively, into \( (6) \) leads to

\[
\frac{Y_{1}}{2\pi \sin\theta} = \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta) d\theta - \frac{i}{Z_{0}} \sum_{\nu_{p}} \frac{\partial}{\partial \theta} T_{\nu_p}(\theta) \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} T_{p}(\cos\theta) d\theta
\]

\[
\times \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} T_{p}(\theta) d\theta
\]

\[
\times \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} P_{r}(\cos\theta) d\theta
\]

the integral equation for the unknown aperture field \( E_{a}(\theta) \).

Next, multiply both sides of \( (17) \) by \( E_{a}(\theta) \sin\theta \) and integrate with respect to \( \theta \) from \( \psi_{1} \) to \( \psi_{2} \) to get

\[
Y_{1} = \frac{2\pi}{Z_{0}} \left[ \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta) d\theta \right]^2 \times \left[ \sum_{\nu_{p}} (\nu_{p}(\nu_{p}+1)N_{\nu_p} I_{p} \nu_{p})^{-1} \times \left\{ \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} T_{p}(\theta) d\theta \right\}^2 \right.
\]

\[
- \sum_{s=1}^{\infty} \frac{\Omega(\psi_{2})}{\nu_{s}}(s+1)M_{s}I_{s}\nu_{s}^{-1} \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} P_{r}(\cos\theta) d\theta \]

\[
\times \left. \left\{ \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} P_{r}(\cos\theta) d\theta \right\}^2 \right].
\]

It can be shown that the functional \( Y_{1} \) of \( (18) \) is in variational form with respect to \( E_{a}(\theta) \); that is, \( E_{a} \) makes \( Y_{1} \) stationary. In other words, the first variation of \( Y_{1} \) is zero for all admissible variations of \( E_{a} \).

Expand \( E_{a} \) in the form

\[
E_{a}(\theta) = \frac{A_{0}}{\sin\theta} + \sum_{\nu_{m}} A_{\nu_{m}} \frac{\partial}{\partial \theta} T_{\nu_{m}}(\theta)
\]

which is implied by \( (8b) \), where \( \{A_{\nu_{m}}\} \) are unknown constant. Since \( E_{a}(\theta) \) appears in both the numerator and the denominator of the right side of \( (18) \), normalize \( A_{0} \) to unity.

After carrying out the integrals in \( (18) \) using \( (19) \), \( Y_{1} \) becomes

\[
Y_{1} = \frac{iZ_{0}}{2\pi Z_{0}^{2}} \left[ \sum_{s=1}^{\infty} \frac{\Omega(\psi_{2})}{\nu_{s}} \frac{\nu_{s}^{2}}{(s+1)M_{s}I_{s}\nu_{s}} \right.
\]

\[
+ \sum_{\nu_{m}} \frac{\nu_{m}(\nu_{m}+1)I_{\nu_{m}}}{N_{\nu_{m}}} \nu_{m}^{-1} \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} T_{\nu_{m}}(\theta) d\theta
\]

\[
- \sum_{s=1}^{\infty} \frac{\nu_{s}^{2}}{(s+1)M_{s}I_{s}\nu_{s}} \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} P_{r}(\cos\theta) d\theta
\]

\[
\times \left. \left[ \sum_{s=1}^{\infty} \frac{\Omega(\psi_{2})}{\nu_{s}}(s+1)M_{s}I_{s}\nu_{s}^{-1} \int_{\psi_{1}}^{\psi_{2}} E_{a}(\theta)\sin\theta \frac{\partial}{\partial \theta} P_{r}(\cos\theta) d\theta \right] \right].
\]

To simplify \( (20) \), introduce \( \beta_{\nu_m}, \alpha_{\nu_m}, \) and \( \gamma_{\nu_m,\nu_m} \) as follows:

\[
\beta_{\nu_m} = \frac{iZ_{0}}{2\pi Z_{0}^{2}} \nu_{m}(\nu_{m}+1)\frac{I_{\nu_m}}{N_{\nu_m}},
\]

\[
\alpha_{\nu_m} = \frac{iZ_{0}}{2\pi Z_{0}^{2}} \nu_{m}^{2}(\nu_{m}+1)\frac{I_{\nu_m}}{M_{s}I_{s}\nu_{s}},
\]

\[
\gamma_{\nu_m,\nu_m} = \frac{iZ_{0}}{2\pi Z_{0}^{2}} \nu_{m}^{2}(s+1)M_{s}I_{s}\nu_{s}. \]

With \( (16) \) and \( (21) \), \( Y_{1} \) can be cast in the form

\[
Y_{1} = Y_{0} + \sum_{\nu_{m}} \beta_{\nu_m} A_{\nu_m}^{2} + 2 \sum_{\nu_{m}} \alpha_{\nu_m} A_{\nu_m}
\]

\[
+ \sum_{\nu_{m}} \gamma_{\nu_m,\nu_m} A_{\nu_m} A_{\nu_m},
\]

Since \( Y_{1} \) is stationary with respect to the variation of \( E_{a} \), one determines the unknown coefficients \( A_{\nu_m} \) by setting \( \frac{\partial}{\partial A_{\nu_m}} Y_{1} = 0 \) which gives

\[
\beta_{\nu_m} A_{\nu_m} + \alpha_{\nu_m} + \sum_{\nu_{m}} \gamma_{\nu_m,\nu_m} A_{\nu_m} = 0.
\]

Multiply \( (22b) \) by \( A_{\nu_m} \), sum over \( \nu_{m} \), and substitute the resulting expression into \( (22a) \) to obtain

\[
Y_{1} = Y_{0} + \sum_{\nu_{m}} \alpha_{\nu_m} A_{\nu_m}.
\]

If the first term on the right side of \( (23) \) is called the zero-order solution, then the second expression may be called the correction term.
C. Wide-Angle Approximation

As stated earlier, in the wide-angle approximation, the complementary waves (TM modes) in the antenna region are negligible ($\alpha_{\phi} \approx 0$). This approximation corresponds to $\psi_1$ and $\psi_2$ that exceed $40^\circ$. Consequently, for $0 < r < a$ the field components can be expressed as

$$E_\theta = \frac{Z_0}{r \sin \theta} \left[ \sigma_1 e^{-ikr} - \sigma_2 e^{ikr} \right]$$

and

$$H_\phi = \frac{1}{r \sin \theta} \left[ \sigma_1 e^{-ikr} + \sigma_2 e^{ikr} \right]$$

(24)

where

$$\sigma_1 = \frac{I_0}{2\pi} (Z_1 Y_1 + 1) e^{ika}$$

$$\sigma_2 = \frac{I_0}{2\pi} (Z_1 Y_1 - 1) e^{-ika}$$

$$\sigma_2 = \frac{Z_0 Y_1 - 1}{Z_0 Y_1 + 1} e^{-2ika}$$

(25)

The ratio $\sigma_2/\sigma_1$ is the reflection coefficient.

In the exterior region ($r > a$)

$$E_\theta = iZ_0 \sum_{n=1}^{\infty} B_n \left[ h_n^{(2)}(kr) - \frac{n}{kr} h_n^{(2)}(kr) \right] P_n^1(\cos \theta)$$

and

$$H_\phi = \sum_{n=1}^{\infty} B_n h_n^{(2)}(kr) P_n^1(\cos \theta)$$

(26)

where

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_n^{(2)}(x)$$

$$B_n = \frac{b_n}{2\pi an(n+1)h_n^{(2)}(ka)}$$

(27a)

$$P_n^1(\cos \theta) = \frac{\sin \theta}{n} \left[ P_{n-1}(\cos \theta) - \cos \theta P_n(\cos \theta) \right]$$

(27b)

The prime in $P_n'$ denotes differentiation with respect to its argument, and the function $P_n^1$ is the associated Legendre function. One can show that the input current $I(0)$ (which is not $I_0$) and the input voltage $V(0)$ are given by

$$I(0) = \lim_{r\to0} \left[ 2\pi r \sin \psi_2 H_\phi(r, \psi) \right] = 2\pi (\sigma_1 + \sigma_2)$$

$$V(0) = \int_{\psi_1}^{\psi_2} r E_\theta(r, \theta) d\theta = 2\pi Z_c (\sigma_1 - \sigma_2).$$

(28a)

Since

$$I_0 = \pi (\sigma_1 e^{-ika} - \sigma_2 e^{ika})$$

$$I(0) = \frac{I_0}{2[Z_c Y_1 \cos(ka) + i \sin(ka)]}$$

(28b)

clearly $I(0)$ is not necessarily equal to $I_0$. Moreover, the input impedance $Z_{in}$ is given by

$$Z_{in} = \frac{V(0)}{I(0)} = Z_c \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}.$$  

(29)

After setting $\alpha_{\phi} = 0$ in (9) and using (27a), one determines that

$$B_n = \frac{-i}{2n(n+1)} \left( \sigma_1 e^{-ika} - \sigma_2 e^{ika} \right) g_n(\mu_1 \mu_2) \Omega(\psi_2)$$

$$h_n^{(2)}(ka) - \frac{n}{ka} h_n^{(2)}(ka).$$

(30)

The substitution of $B_n$ into (26) gives the field components in the exterior region ($r > a$). For observation points such that $kr \gg 1$, an asymptotic approximation of $h_n^{(2)}$ is

$$h_n^{(2)}(kr) \sim e^{ikr}/kr.$$  

(31)

Hence, the radiated field

$$E_{rad}(r, \theta, \omega) = \frac{iZ_0}{2\pi} e^{-ikr} I(0)[k h_e(\theta, \omega)]$$

(32a)

is represented in terms of the antenna’s effective height $h_e$, where

$$k h_e(\theta, \omega) = \frac{e^{-ika} - \sigma_2 e^{ika}}{\sigma_1 + \sigma_2} \sum_{n=1}^{\infty} \frac{i^{n-1}(2n+1)}{n(n+1)} 2n(n+1)$$

$$\times \frac{P_n^1(\cos \theta) g_n(\mu_1 \mu_2) \Omega(\psi_2)}{h_n^{(2)}(ka) - \frac{n}{ka} h_n^{(2)}(ka)},$$

(32b)

Additional information is now used to derive an alternate representation for the coefficient preceding the summand in (32b) and to obtain $Y_1$ under the wide-angle approximation. To these ends, equate the expressions in (24) and (26) at $r = a$ and use (2) and (30). After manipulation, one obtains the relation

$$1 + \frac{\sigma_2 e^{2ika}}{\sigma_1} = -\frac{iS}{4\pi Z_c Z_o} = -i S$$

(33a)

where $S = \hat{S} Z_o/(4\pi Z_c)$

$$\hat{S} = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} g_n(\mu_1 \mu_2) g_n(ka) \Omega(\psi_2)$$

(33b)

Solving (33a) for $\sigma_2/\sigma_1$ leads to

$$\frac{\sigma_2}{\sigma_1} = e^{-2ika} \frac{i S + 1}{i S - 1}.$$  

(34a)

On comparing (25) and (34a), one arrives at

$$Y_1 = \frac{iS}{Z_c} = \frac{1 + \frac{\sigma_2 e^{2ika}}{\sigma_1}}{Z_c 1 - \frac{\sigma_2 e^{2ika}}{\sigma_1}}$$

(34b)

for the terminal admittance.

Equations (32) and (34) provide expressions for the field, the effective height, and the terminal admittance under the wide-angle approximation. In the next two sections, these results are simplified for electrically small ($ka \ll 1$) and electrically large ($ka \gg 1$) wide-angle bicones.
1) Small-Cone Approximation \((ka \ll 1)\): In this case, consider \(\zeta_n\) of (33b) first. Through (27a), express \(\zeta_n\) in terms of the Bessel functions of the first \((J_n)\) and second \((Y_n)\) kinds and substitute the standard asymptotic approximations of \(J_n(ka)\) and \(Y_n(ka)\) for the small argument \(ka\) [10]. In the resulting asymptotic approximation of \(\zeta_n\), one eliminates terms with \((ka)^n\) for \(n \geq 2\) and expands the remaining expression in a series involving powers of \((ka)\) with the binomial expansion. Finally, retaining only terms of the expansion through \((ka)^2\) inclusive leads to the approximation

\[
\zeta_n(ka) \simeq -\frac{ka}{n} \left[ 1 + \frac{(ka)^2}{n(2n-1)} \right].
\] (35a)

Next, substitute (35a) into (33b) to get

\[
S \simeq -\frac{Z_0ka}{4\pi Z_e} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} g_n^2(\mu_1, \mu_2) \Omega(\psi) + \mathcal{O}\left(\frac{(ka)^3}{2n^3}\right).\] (35b)

Now an approximation of \(h_{he}\) is obtained. One can argue that

\[
h_{n-1}^{(2)}(ka) \simeq \frac{2^n(n-1)!}{2(2n+1)!} (ka)^{n+2}
\]

\[-\frac{1}{2^n(2n+1)} + i2^n \frac{(n+1)(n+1)!}{(2n+1)(2n+2)!} (ka)^{2n+1}.\] (36a)

Moreover, substituting (35b) into (34a) and expanding the trigonometric functions in Maclaurin series yield

\[
e^{-ika} \frac{-\sigma_\psi e^{\psi a}}{ka(1+\frac{\sigma_\psi}{\sigma})} \simeq -\frac{i}{(ka)^2} \left[ 1 + \frac{Z_0 \Omega(\psi)}{4\pi Z_e(\psi_1, \psi_2)} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} g_n^2(\mu_1, \mu_2) \right]
\]

\[-\frac{1}{6} + \frac{Z_0 \Omega(\psi)}{8\pi Z_e(\psi_1, \psi_2)} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} g_n^2(\mu_1, \mu_2) + \mathcal{O}((ka)^3)\]. (36b)

Substituting (36) into (32b) implies that

\[
k_{he}(\theta, \omega) \simeq \frac{N(\theta, \omega)}{D(\theta, \omega)}\] (37a)

\[
N(\theta, \omega) = -k_0 \Omega(\psi) \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} (2n+1)(n+1)! \]

\[
\times \frac{P_2^n(\cos\theta)\sigma_\psi(\mu_1, \mu_2)(ka)^{n-1}}{1 - \frac{(ka)^2}{2^n(2n+1)} + 2\frac{2n+1}{2n+2}(\frac{2n+1}{2n+2})^{2n+1}(ka)^{2n+1}}.\] (37b)

As expected, the input impedance of an electrically small bicone behaves like a capacitive impedance \(1/(i\omega C_{in})\), where the equivalent capacitance \(C_{in}\) is computed by making the obvious identification in either of (39).

An identity between two evaluations—\(Y_0(\psi_1, \psi_2)\) and \(Y_0(\psi_1, \psi_2)/2\)—of the zero-order approximation of the terminal admittance is now established. When either \(\psi_2 = \pi/2\) or \(\psi_2 = \psi_1\), the boundary conditions impose the constraint that

\[
\frac{Z_{in}}{i\omega C_{in}} \simeq \frac{1}{1 + \frac{Z_0 g(\psi_2)}{4\pi Z_e(\psi_1, \psi_2)} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} (ka)^{2n+1}} + \mathcal{O}((ka)^4)
\]

\[
D(\theta, \omega) = \left[ 1 + \frac{Z_0 \Omega(\psi)}{4\pi Z_e(\psi_1, \psi_2)} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} g_n^2(\mu_1, \mu_2) \right]
\]

\[-\frac{1}{6} + \frac{Z_0 \Omega(\psi)}{8\pi Z_e(\psi_1, \psi_2)} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)} g_n^2(\mu_1, \mu_2) + \mathcal{O}((ka)^3)\]. (37c)
the index \( r \) of the summation in (15) runs over the odd natural numbers only. Since \( P_r(0) = 0 \) for odd \( r \)

\[
Y_{\ell 0}(\psi_1, \psi_2)Z_{\ell c}(\psi_1, \psi_2) = 2Y_{\ell 0}(\psi_1, \pi/2)Z_{\ell c}(\psi_1, \pi/2).
\]

(40a)

However, \( Z_{\ell c}(\psi_2, \pi/2) = 2Z_c(\psi_2, \pi) = (Z_0/\pi) \ln[\cot(\psi_2/2)] \)

by (2a), which implies that

\[
Y_{\ell 0}(\psi_1, \pi/2) = \frac{1}{2} Y_{\ell 0}(\psi_1, \pi/2).
\]

(40b)

Note that (40) and relations based on it do not depend on the small-cone approximation.

In summary, observe that the radiated field and the input impedance for the electrically small wide-angle bicone depend explicitly on the cone angles \( \psi_1 \) and \( \psi_2 \). Moreover, upon setting \( \psi_1 = 0, \psi_2 = \pi/2, \) and \( \theta = \pi/2 \), \( k h_e \) in (38b) and \( Z_{\text{in}} \) in (39b) reduce to

\[
k h_e \simeq \frac{6\pi k a Z_c \cos \theta_0}{4\pi Z_c + 3Z_0 \cos^2 \theta_0}
\]

and

\[
Z_{\text{in}} \simeq \frac{4\pi Z_c^2}{i\kappa(4\pi Z_c + 3Z_0 \cos^2 \theta_0)}
\]

(41)

which are [9, Eq. (10)] and [11, Eq. (22)], respectively.

2) Large-Cone Approximation \((k\alpha \gg 1)\): In the high-frequency region, expressions for \( \zeta_b \) and \( \sigma_2/\sigma_1 \) must be established to derive results for \( k h_e \) and \( Z_{\text{in}} \). The analysis of this section extends the approach of [9, Appendices B and C] to obtain the desired results for the more general situation of arbitrary \( \psi_1 \) and \( \psi_2 \). In particular, an expression for \( \sigma_2/\sigma_1 \) is obtained by using identities involving the Legendre polynomials.

According to [12]

\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} P_{\ell}^{(\mu_1)} P_{\ell}^{(\mu_2)} = 2\ln 2 - 1 - \ln[(1 - \mu_1)(1 + \mu_2)]
\]

(42a)

for \(-1 < \mu_1 \leq \mu_2 < 1\) since \( \mu_1 \neq \pm 1 \) and \( \Omega(\psi_2) = 1 \).

Replacing \( \mu_1 \) with \(-\mu_1 \) in (42a) and noting that \( P_{\ell}(-\mu_1) = (-1)^{\ell} P_{\ell}^{(\mu_1)} \) imply

\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} (-1)^{\ell} P_{\ell}^{(\mu_1)} P_{\ell}^{(\mu_2)} = 2\ln 2 - 1 - \ln[(1 + \mu_1)(1 + \mu_2)]
\]

(42b)

for \(-1 < -\mu_2 \leq \mu_1 < 1\). Set \( \mu_2 = -\mu_1 \) in (42b) to get

\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} P_{\ell}^{(\mu_1)} P_{\ell}^{(\mu_2)} = 2\ln 2 - 1 - \ln(1 - \mu_1^2)
\]

(42c)

for \(-1 < \mu_1 < 1\). Consequently, by (2a), (7b), (42b), and (42c), one obtains

\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} g^{(\mu_1, \mu_2)} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} [\tilde{P}_{\ell}^{(\mu_1)} - 2(-1)^{\ell} P_{\ell}^{(\mu_1)} P_{\ell}^{(\mu_2)} + P_{\ell}^{(\mu_2)}]
\]

\[
= 2\ln \left[ \frac{\cot(\psi_1/2)}{\cot(\psi_2/2)} \right]
\]

\[
= \frac{4\pi}{Z_0} Z_c(\psi_1, \psi_2)
\]

(43)

for \(-1 < -\mu_1 \leq \mu_2 < 1\). Since \( 0 < \psi_1 < \pi/2 \) and \( -\cos(\psi_1) < \cos(\psi_2) \) for any \( \psi_1 \) and \( \psi_2 \), the condition \(-1 < -\mu_1 \leq \mu_2 < 1\) is satisfied. Hence, (43) is valid for the biconical geometry.

Before addressing \( \sigma_2/\sigma_1 \), an asymptotic approximation for \( \zeta_b \) is developed. By (31), one may argue for \( I/(k\alpha) < 1 \) that

\[
\zeta_b(k\alpha) \sim \frac{h_{\text{out}}^{(2)}(k\alpha)}{h_{\text{in}}^{(2)}(k\alpha)} = \left[ \frac{l^2}{l^2(k\alpha)} \right]^{-1} \sim i \sum_{r=0}^{\infty} \left( \frac{l}{k\alpha} \right)^r \sim i + O\left( \frac{l}{k\alpha} \right).
\]

(44a)

Thus, for fixed \( l \)

\[
\lim_{k\alpha \to \infty} \zeta_b(k\alpha) = i.
\]

(44b)

From (33b), (34a), and (44a), note that \( \sigma_2/\sigma_1 \) has two infinite sums where the index \( l \) runs from one to \( \infty \). Consequently, one may not simply substitute the asymptotic result for \( \zeta_b \) into (34a) to get \( \sigma_2/\sigma_1 \) because \( l \) is not fixed. However, by taking advantage of the convergence of the two series

\[
\sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} g^{(\mu_1, \mu_2)}(k\alpha) \quad \text{and} \quad \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} g^{(\mu_1, \mu_2)}(k\alpha)
\]

(45a)

and by using (44b), one may argue with mathematical rigor that

\[
\lim_{k\alpha \to \infty} \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{(\ell + 1)^2} g^{(\mu_1, \mu_2)}(k\alpha) \sim i \sum_{r=0}^{\infty} \left( \frac{l}{k\alpha} \right)^r \sim i + O\left( \frac{l}{k\alpha} \right)
\]

and

\[
\lim_{k\alpha \to \infty} \frac{iS + 1}{iS - 1} = 0 = \lim_{k\alpha \to \infty} \frac{\sigma_2}{\sigma_1}.
\]

(45b)

To determine the behavior of the effective length of the wide-angle bicone for \( k\alpha \gg 1 \), one must evaluate two more summations involving the Legendre polynomials. First, let \( \mu_2 = \mu = \cos \theta \) in (42a); second, interchange \( \mu_1 \) and \( \mu_2 \) in (42a) and let \( \mu_2 = \mu \); third, let \( \mu_1 = \mu \) in (42b); and fourth, interchange \( \mu_1 \) and \( \mu_2 \) in (42b), replace \( \mu_1 \) with \(-\mu_1 \) and let...
$\mu_1 = \mu$. Performing these manipulations leads to

$$
\sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} P_l(\mu) P_l(\mu_1)
= \begin{cases}
2\ln 2 - 1 - \ln(1 - \mu)(1 + \mu), & -1 < \mu \leq \mu_1 \\
2\ln 2 - 1 - \ln(1 - \mu_1)(1 + \mu_1), & \mu_1 \leq \mu < 1.
\end{cases}
$$

(46a)

$$
\sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} (-1)^l P_l(\mu) P_l(\mu_2)
= \begin{cases}
2\ln 2 - 1 - \ln(1 - \mu_2)(1 - \mu), & -1 < \mu \leq -\mu_2 \\
2\ln 2 - 1 - \ln(1 + \mu_2)(1 + \mu_2), & -\mu_2 \leq \mu < 1.
\end{cases}
$$

(46b)

After differentiating (46) with respect to $\mu$, one obtains

$$
\sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \frac{d}{d\mu} P_l(\mu) P_l(\mu_1)
= \begin{cases}
\sqrt{\frac{1+\mu}{1-\mu}}, & -1 < \mu \leq \mu_1 \\
\sqrt{\frac{1-\mu}{1+\mu}}, & \mu_1 \leq \mu < 1.
\end{cases}
$$

(47a)

$$
\sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} (-1)^l \frac{d}{d\mu} P_l(\mu) P_l(\mu_2)
= \begin{cases}
\sqrt{\frac{1+\mu}{1-\mu}}, & -1 < \mu \leq -\mu_2 \\
\sqrt{\frac{1-\mu}{1+\mu}}, & -\mu_2 \leq \mu < 1.
\end{cases}
$$

(47b)

The last ingredient necessary for calculating the limit of $kh_e$ is the asymptotic result

$$
h_{l(0)}^{(2)}(ka) = \frac{l}{ka} h_{l(0)}^{(2)}(ka) \sim \frac{1}{ka} e^{-ka} \left[1 - \frac{l}{ka}\right]
$$

(48)
as $ka \to \infty$. Equation (48) implies

$$
\lim_{ka \to \infty} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \frac{d^{-1} e^{-i\alpha} P_l^2(\mu) [\eta_l(\mu_1, \mu_2)]}{ka \left[h_{l(0)}^{(2)}(ka) - \frac{l}{ka} h_{l(0)}^{(2)}(ka)\right]} = -i \frac{2l+1}{l(l+1)} P_l^2(\mu) [\eta_l(\mu_1, \mu_2)],
$$

(49)

Substituting (45d) and (49) into (32b) and using (47) yield

$$
\lim_{ka \to \infty} kh_e(\theta, \omega)
= \lim_{ka \to \infty} \frac{1 - \sigma_\alpha e^{2\alpha k a}}{1 + \sigma_\alpha} \frac{\Omega(\psi_2)}{2}
\times \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \frac{d^{-1} e^{-i\alpha} P_l^2(\mu) [\eta_l(\mu_1, \mu_2)]}{ka \left[h_{l(0)}^{(2)}(ka) - \frac{l}{ka} h_{l(0)}^{(2)}(ka)\right]} = -\frac{i}{2} \frac{2l+1}{l(l+1)} P_l^2(\mu) [\eta_l(\mu_1, \mu_2)] = -\frac{i}{2} \frac{\Omega(\psi_2)}{2}
$$

$$
\times \begin{cases}
\sqrt{\frac{1+\mu}{1-\mu}}, & -1 < \mu \leq -\mu_2 \\
\sqrt{\frac{1-\mu}{1+\mu}}, & -\mu_2 \leq \mu < 1
\end{cases}
$$

(50)

This limit implies that the behavior of the field $|E_\theta|$ for $\psi_1 < \theta < \pi - \psi_2$ approaches $1/\sin \theta$ as the frequency increases without bound and that the directions of maximum radiation approach $\psi_1$ and $\pi - \psi_2$. Consequently, maximum radiation of the electrically large wide-angle bicone does not occur at broadside ($\theta = \pi/2$).

Furthermore, with (29) and (45c), one can easily show that the bicone’s input impedance has the limit

$$
\lim_{ka \to \infty} Z_{in} = Z_0(\psi_1, \psi_2).
$$

(51)

Thus, for high frequencies the input impedance is essentially constant. Hence, the electrically large wide-angle bicone is a broadband and possibly an ultrawide-band antenna.

3) Examples: Two special cases are considered for $a = 20$ in: 1) $\psi_1 = 53.1^\circ = \psi_2$ and 2) $\psi_1 = 53.1^\circ$ and $\psi_2 = 70^\circ$. The relative field pattern associated with the $E$-plane radiation pattern [7]

$$
R(\theta, \omega) = \frac{E_{\text{rad}}^\text{in}(r, \theta, \omega)}{E_{\text{rad}}^\text{in}(r, \pi/2, \omega)}
$$

$$
= \sum_{n=1}^{\infty} \frac{\eta^{n-1}(2n+1)}{2n(n+1)} \frac{P_k^2(\cos \theta) P_0^2(\sin \theta)}{h_{l(0)}^{(2)}(ka) - \frac{l}{ka} h_{l(0)}^{(2)}(ka)}
$$

(52)
is plotted for various values of $ka$ in both cases.

In case 1), as mentioned previously, only the odd terms of the series are present because $\psi_1 = \psi_2$. Hence, on re-indexing each sum in (52) by setting $n = 2m + 1$ and by allowing $m$ to run from zero to $\infty$, (52) becomes [8, Eq. (6)]. Truncate each series of the re-indexed version of (52) at $M + 1$ terms and denote the resulting fraction as $R_M$. The approximate pattern $|R_M|$ is plotted in Fig. 3 for five frequencies ranging from $ka = 1$ to $ka = 0.1$. The correspondence between $ka$ and frequency $f$ for each plot is shown in Table I. One high frequency and one low frequency are chosen to represent situations when $ka \ll 1$ and $ka \gg 1$, respectively; while the other three frequencies are selected for comparison to [8, Fig. 5(c)], as well as for providing nominal values in the transition between $ka \ll 1$ and $ka \gg 1$. The pattern $|R_M|$ has azimuthal symmetry and symmetry about $\theta = \pi/2$. The former means that the pattern may be graphed in two dimensions and the latter means that one may restrict $\theta$ to $[0, \pi/2]$ to gain a complete understanding of the behavior of $R_M$.

As one can observe [Fig. 3(a)], the pattern at low frequencies is similar to that of the short dipole as [8] notes. In fact, on utilizing the low-frequency approximation in (38b), one finds that $R \sim \sin \theta$, which is the short-dipole pattern. The graph of $R \sim \sin \theta$ is coincident with that of Fig. 3(a) when they are

<table>
<thead>
<tr>
<th>Figure</th>
<th>$f$</th>
<th>$ka$</th>
<th>Range of $ka$</th>
<th>$M$ Used</th>
<th>$M$ Needed</th>
</tr>
</thead>
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<td>1 MHz</td>
<td>0.0106</td>
<td>$ka &lt; 1$</td>
<td>29</td>
<td>3</td>
</tr>
<tr>
<td>3b</td>
<td>100 MHz</td>
<td>1.0640</td>
<td>Transitional</td>
<td>29</td>
<td>3</td>
</tr>
<tr>
<td>3c</td>
<td>500 MHz</td>
<td>5.3198</td>
<td>Transitional</td>
<td>29</td>
<td>3</td>
</tr>
<tr>
<td>3d</td>
<td>1 GHz</td>
<td>10.6356</td>
<td>Transitional</td>
<td>29</td>
<td>6</td>
</tr>
<tr>
<td>3e</td>
<td>10 GHz</td>
<td>106.3560</td>
<td>$ka &gt; 1$</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>3f</td>
<td>10 GHz</td>
<td>106.3560</td>
<td>$ka &gt; 1$</td>
<td>29</td>
<td>60</td>
</tr>
</tbody>
</table>

TABLE I

Correspondence Among Upper Limit $M$ of Approximate Radiation Distribution Function $R_M$, $ka$, and Frequency $f$ in Fig. 3.
Fig. 3. Approximate magnitude $|R_{x,x}|$ of the normalized radiation distribution of $E_\theta$ ($E$-plane radiation pattern) for a spherically capped biconical antenna with equal cone angles in free space. The cone angles $\psi_1$ and $\psi_2$ are equal to $53.1^\circ$, and the slant height (height) is 20 in (12 in). (a) $ka = 0.0106$. (b) 1.0640. (c) 5.3198. (d) 10.6395. (e) 106.3950. (f) $|R_{x,z}|$ is graphed less accurately for $ka = 106.3950$ by using only 30 terms ($M = 29$) of each series in (52).

overlayed. Even the pattern for the transitional value of $ka = 1.0640$ is only slightly distinguishable from the short dipole’s pattern. Therefore, the approximation for $ka \ll 1$ is excellent for $ka \leq 0.2$, and is good for $0.2 < ka \leq 1.06$. As $ka$ increases from 1 to 10.640 through the transitional region between electrically small and electrically large bicones [Fig. 3(b)–(d)], the peak radiation remains at broadside; however, the radiation decreases for $18^\circ \lesssim \theta < 90^\circ$, and for each $\theta$ between $0^\circ$ and $18^\circ$ the radiation level exceeds that of the short dipole with a local maximum appearing at some intermediate angle. Fig. 3(b)–(d) agrees well with [8, Fig. 5(c)]. Sandler and King don’t indicate how many terms of the series they use to generate their figures; however in the Mathematica calculations used to generate Fig. 3, one obtains good graphical depictions for $ka = 1.0640, 5.3198, 10.6395$ when $M = 3, 3, 6$, respectively. As Table I indicates, Fig. 3(a)–(d) is generated for 30 terms of each series ($M = 29$). This value of $M$ is picked because each pattern is accurate for $ka \leq 10.6395$ and because Harrison used this value for calculating tables in [11].

In contrast, to obtain a reasonably accurate pattern for $ka = 106.3950$, at least 61 terms ($M = 60$) must be used [Fig. 3(e)]. Fig. 3(f) plots $ka = 106.3950$ for 30 terms and is provided as a comparison to Fig. 3(e). Clearly, the detail is missing in Fig. 3(f). The pattern in Fig. 3(e) is considerably less smooth than the patterns for small and transitional values of $ka$. In particular, many smaller amplitude lobes appear for $0^\circ < \theta \leq 30^\circ$. The effect is less pronounced between $30^\circ$ and $90^\circ$, where the pattern has a somewhat wavy nature as
Fig. 4. $E$-plane radiation pattern in the high-frequency limit ($ka \to +\infty$) for a spherically capped biconical antenna with equal cone angles in free space. The cone angles $\psi_1$ and $\psi_2$ are equal to $53.1^\circ$ and the slant height (height) is 20 in (12 in).

Evidenced by the five local maxima and four local minima. The lobe structure for small $\theta$ is not unexpected because the number of oscillations and the magnitude near $\theta = 0$ of the Associated Legendre Function $P^{3}_{m+1}(\cos \theta)$ increase as $m$ increases. Consequently, as $ka$ increases, $M$ must increase to get accurate results. One major feature that distinguishes the pattern of a wide-angle bicone for high frequencies from the patterns at other frequencies is the migration of the direction of peak radiation away from broadside. In Fig. 3(e), the peak occurs well off broadside at $\theta = 63.5^\circ$. As $ka$ increases, one expects the peak to approach the cone's surface at $\psi_1 = 53.1^\circ$ since by (50) and (52), $R(\theta, \omega) \to 1/\sin \theta$ as $ka \to \infty$ for $\psi_1 < \theta \leq \pi/2$. The limiting value in Fig. 4 is the triangular region bounded by $R = 1/\sin \theta$ (the vertical line segment $y = 1$), the line $z = 0$, and the line $z = y \cot(53.1^\circ)$. Consequently, as $ka \to \infty$, the pattern calculated with (52) will approach the pattern of Fig. 4. Fig. 3(e) is consistent with Fig. 4 in that most of the energy is radiated in $\psi_1$ and the five local maxima are very close to the vertical line $y = 1$.

In case 2), the summations in (52) are truncated at $N$ terms and the resulting fraction is denoted $R_N$. Although not shown, the pattern looks like that of the short dipole for $ka \leq 1$. As $ka$ increases from unity through transitional values, the effect of distinct bicone angles is manifested in asymmetrical patterns relative to the $y$-axis. Specifically, in the limit as $ka \to +\infty$, $|R_N| \to 1/\sin \theta$. Moreover, the pattern approaches the triangular region bounded by the lines $R = 1/\sin \theta$, $z = -y \cot(70^\circ)$, and $z = y \cot(53.1^\circ)$, which are indicated by the dashed line segments in Fig. 5. To illustrate the behavior for large $ka$, Fig. 5 also displays the pattern for $ka = 106.3950$ and $N = 120$ (solid curve). This pattern has many small amplitude lobes for $0^\circ \leq \theta \leq 50^\circ$ and $110^\circ \leq \theta \leq 180^\circ$. For angles between $60^\circ$ and $108^\circ$, several relative maxima and minima occur near the line $y = 1$, with the two largest maxima at $\theta = 63.8^\circ$ and $90.7^\circ$. As $ka$ and $N$ increase, the number of these local extrema increases and the associated $|R_N|$ approaches the line $y = 1$ with the largest and second largest values of $|R_N|$ approaching the edges of the upper and lower cones, respectively. The two obvious differences from case 1) are: a) the loss of symmetry in the pattern about the $y$ axis of case 2) and b) the peak radiation occurs near the cone with the smaller cone angle in case 2).

These examples provide analytical justification of Sandler and King’s numerically based observations [8] that the wide-angle bicone behaves like a short dipole at low frequencies (38b) and has its direction of maximum radiation moving away from broadside toward the bicone’s surface for high frequencies (50). Consequently, the tacit assumption in [9, Appendixes B and C] that the peak radiation occurs at $\theta = \pi/2$ for electrically large bicones is not justified. Since an ultrawide-band input signal’s passband [13] may contain both low- and high-frequency spectral components relative to the bicone’s passband, the explicit analytical characterizations of the frequency-dependent behavior contained herein are essential in analyzing radiation of ultrawide-band signals.

III. TIME HISTORY OF RADIATED AND RECEIVED FIELDS ASSOCIATED WITH ELECTRICALLY SMALL AND ELECTRICALLY LARGE WIDE-ANGLE BICONES

Thus far, the discussion of electrically small and large wide-angle conical antennas has concentrated on the impact to quantities in the frequency domain. Of interest, especially when the input is wide-band or ultrawide-band, is the temporal behavior of the radiated field and received voltage. This section expresses the time-domain radiated field of wide-angle conical antennas in terms of the input voltage and expresses the time-dependent load voltage of a receiving conical antenna in terms of the incident field. In particular, four special cases are considered: 1) transmission for $ka \ll 1$; 2) transmission for $ka \gg 1$; 3) reception for $ka \ll 1$; and 4) reception for $ka \gg 1$. 
Let $V_g(t)$ be the input source voltage of the transmitting antenna, and let $\hat{V}_g(\omega)$ be the Fourier transform of $V_g(t)$. If $Z_{in}(\omega)$ and $Z_g(\omega)$ are the input impedance of the antenna and the generator (source) impedance, respectively, the input current $I(0)$ (or $I(0,\omega)$) and $\hat{V}_g(\omega)$ are related by

$$I(0) = \frac{\hat{V}_g(\omega)}{Z_{in}(\omega) + Z_g(\omega)}. \quad (53)$$

Substituting (53) into (32a) yields

$$E_{\theta}^{rad}(r,\theta,\omega) = \frac{iz_0}{2\pi} \left( \frac{\hat{V}_g(\omega)}{Z_{in}(\omega) + Z_g(\omega)} \right) \frac{e^{-i\omega r}}{r} k_{he}(\theta,\omega). \quad (54)$$

Consequently, the transfer function $T(r,\theta,\omega)$ of the transmitting antenna is defined by

$$T(r,\theta,\omega) = \frac{E_{\theta}^{rad}(r,\theta,\omega)}{\hat{V}_g(\omega)} = \frac{i\omega Z_0}{c r} k_{he}(\theta,\omega) e^{-i\omega r/c} \left( \frac{Z_{in}(\omega) + Z_g(\omega)}{Z_{in}(\omega) + Z_g(\omega)} \right). \quad (55)$$

The time-dependent radiated field $e_{\theta}^{rad}(r,\theta,t)$, the inverse Fourier transform of $E_{\theta}^{rad}(r,\theta,\omega)$, is given formally by

$$e_{\theta}^{rad}(r,\theta,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}_g(\omega) T(r,\theta,\omega) e^{i\omega t} d\omega. \quad (56)$$

When the antenna is receiving, define the open-circuit receiving voltage $\hat{V}_{oc}(\omega)$ by

$$\hat{V}_{oc}(\omega) = -E_{\theta}^{oc}(r,\theta,\omega) h_{oc}(\theta,\omega). \quad (57a)$$

In this expression, $\theta$ gives the direction of the incident field relative to the axis of the receiving bicone. If the receiving antenna is connected to a load with impedance $Z_L(\omega)$, then the received voltage $\hat{V}_L$ across this load is

$$\hat{V}_L(\omega) = \frac{E_{\theta}^{oc}(r,\theta,\omega) h_{oc}(\theta,\omega) Z_L(\omega)}{Z_{in}(\omega) + Z_L(\omega)} \quad (57b)$$

where $Z_{in}$ is the input impedance of the antenna and is the same impedance when the antenna is used for transmitting. One can also define the reception transfer function $S(\theta,\omega)$ by

$$S(\theta,\omega) = \frac{\hat{V}_L(\omega)}{E_{\theta}^{oc}(r,\theta,\omega)} = \frac{h_{oc}(\theta,\omega) Z_L(\omega)}{Z_{in}(\omega) + Z_L(\omega)}. \quad (57c)$$

Therefore, the time-dependent load voltage $V_L(t)$ is given formally by

$$V_L(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{V}_{oc}(\omega) Z_L(\omega)}{Z_{in}(\omega) + Z_L(\omega)} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\theta}^{oc}(r,\theta,\omega) S(\theta,\omega) e^{i\omega t} d\omega. \quad (58)$$

When $\hat{V}_g(\omega)$ and $E_{\theta}^{oc}(r,\theta,\omega)$ are known, $e_{\theta}^{rad}(r,\theta,t)$ and $V_L(t)$ can be expressed in closed form either for $k\alpha \ll 1$ or for $k\alpha \gg 1$ under suitable matching conditions. Harrison and Williams [9] considered transient radiation from and reception by wide-angle conical antennas above an infinite conducting plane for various special cases when the observation angle $\theta$ is $\pi/2$. Instead of presenting all the cases enumerated in [9], a few cases are presented for illustrative purposes.

Case 1: Consider radiation from a wide-angle electrically small bicone which is matched to the source ($Z_c = Z_g$). Substituting (39b) for $Z_{in}$ and (38b) for $k_{he}$ in (54) and (56) yields

$$e_{\theta}^{rad}(r,\theta,t) = \frac{3\Omega(\psi_2) Z_0 a^2 (\cos \psi_1 + \cos \psi_2)}{8\pi^2 c^2 r Z_c(\psi_1, \psi_2)} \times \sin \theta \frac{d}{dt} V_g(t - \frac{r}{c}). \quad (59)$$

where $-8\pi Z_0 \gg k\alpha [8\pi Z_c + 3Z_0 \Omega(\cos \psi_1 + \cos \psi_2)^2]$ is satisfied for $k\alpha \ll 1$. Equation (59) shows that for a wide-angle small cone the radiated electric field is proportional to the second time derivative of the retarded input voltage, provided the matching condition $Z_c = Z_g$ is also satisfied. This result is similar to that of a short dipole, which has a spatially invariant current.

Case 2: As a second example, assume the matching condition of Case 1, but let the antenna be electrically large. Equations (50), (51), and (56) lead to

$$e_{\theta}^{rad}(r,\theta,t) = \frac{Z_0}{4\pi r Z_c(\psi_1, \psi_2) \sin \theta} V_g(t - \frac{r}{c}). \quad (60)$$

Thus, for a large wide-angle conical radiator, the field is a retarded replica of the input voltage and is maximized along the conic surface corresponding to the smaller of the two cone angles where the $\sin \theta$ is a minimum. Consequently, this antenna is ideal for ultrawide-band signals with spectra that obey the constraint $k\alpha \gg 1$.

Case 3: In this case, the voltage received by a small bicone that is matched to its load ($Z_L = Z_c$) is obtained. After substituting (39a) for $Z_{in}$ and (38b) for $k_{he}$ in (58), one gets

$$V_L(t) = \frac{3\pi^2}{4c} \Omega(\psi_1 + \cos \psi_2) \cos \psi_1 \cos \psi_2 \times \sin \theta \frac{d}{dt} E_{\theta}^{oc}(r,\theta,\omega) e^{i\omega t} d\omega \quad (61)$$

Hence, the received voltage for a small wide-angle bicone under ideal matching conditions behaves like the temporal derivative of the incident field.

Case 4: Finally, reception by an electrically large wide-angle bicone with a matched load ($Z_L = Z_c$) is considered. Equations (50), (51), and (58) imply

$$V_L(t) = -\frac{c}{2\sin \theta} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\omega} E_{\theta}^{oc}(r,\theta,\omega) e^{i\omega t} d\omega \right] = -\frac{c}{2\sin \theta} \int_{-\infty}^{\infty} e^{i\omega t} d\theta. \quad (62)$$
Clearly, in the preceding cases, the time-dependent radiated field and received load voltage are functions of the input signal’s passband and the matching condition between the antenna and the input’s passband. Cases 1 and 2 indicate that under perfect matching conditions the radiated field has different temporal dependences, as well as the previously stated \( \sin \theta \) and \( 1/\sin \theta \) dependences, for \( k a \ll 1 \) and \( k a \gg 1 \); that is, the field is proportional to the second time derivative of the retarded input voltage and to the retarded input voltage for \( k a \ll 1 \) and \( k a \gg 1 \), respectively. On the other hand, the received load voltage is proportional to the product of \( \sin \theta \) \( (1/\sin \theta) \) and the time derivative (integral) of the incident field for \( k a \ll 1 \) \( (k a \gg 1) \). Furthermore, variations of \( \psi_1 \) and \( \psi_2 \) affect only the amplitudes of the radiated field and received voltage for these cases.

IV. SUMMARY

The results of this study are valid for spherically capped wide-angle biconical antenna configurations, including the degenerate cases of a cone above a ground plane and a bicone with equal cone angles \( \psi_1 = \psi_2 \). In addition, the treatment for arbitrary cone angles \( \psi_1 \) and \( \psi_2 \) subsumes as special cases the results of [1]–[9], [11]. In particular, to the best of the authors’ knowledge, the results for unequal conical angles are new. Exact and approximate expressions for the driving impedance, the effective height, and the radiated field are derived in the frequency domain. Moreover, the approximate expressions derived herein for electrically small \( (k a \ll 1) \) and large \( (k a \gg 1) \) bicones are more accurate than the corresponding ones in the literature. Also, time-domain representations of the radiated field and the received load voltage for \( k a \ll 1 \) and \( k a \gg 1 \) have been derived under different matching conditions. The radiated field and the received voltage vary like \( \sin \theta \) for small wide-angle bicones and like \( 1/\sin \theta \) for large wide-angle bicones.

Sandler and King’s observations regarding the behavior of electrically small (low frequencies) and electrically large (high frequencies) wide-angle bicones are explained by the analytical work of this paper. Furthermore, since the results of Sandler and King (as well as the authors’ results) show that maximum radiation does not take place at broadside \( (\theta = \pi/2) \) for high frequencies, consideration only of the broadside case by Harrison and Williams [9] for \( k a \gg 1 \) is inadequate. In particular, the authors show that maximum radiation occurs along the cone with smaller cone angle. The results contained herein are especially pertinent to ultrawide-band signals, since the passband of such signals may include low, transitional, and high frequencies of a wide-angle bicone’s passband.

Finally, while reproducing some of the numerical results of Sandler and King, a new piece of information has been discovered. Namely, the infinite series for the radiated far field is very slowly convergent for certain high frequencies, which may correspond to the electrical resonances of the conical structure. This aspect of biconical radiation problems requires further detailed investigation.

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REFERENCES

Eric L. Mokole (M’86) was born in Akron, OH, in 1949. He received the B.S. degree in applied mathematics from New York University, NY, in 1971, the M.S. degree in mathematics from Northern Illinois University, DeKalb, in 1973, the M.S. degrees in physics and applied mathematics from the Georgia Institute of Technology, Atlanta, in 1976 and 1978, respectively, and the Ph.D. degree in mathematics from the Georgia Institute of Technology, in 1982.

For the 1982–1983 academic year, he was an Assistant Professor of Mathematics at Kennesaw College, Kennesaw, GA. From 1983 to 1986 he held a position in the Electronic Warfare Division of the Naval Intelligence Support Center (now the Office of Naval Intelligence), Washington, DC. Since 1986 he has been employed by the Radar Division of the Naval Research Laboratory (NRL), Washington, DC. Currently, he is a member of the consulting staff of the Target Characteristics Branch. He has been conducting basic/applied research and system analyses on space-based radar, on shipborne Navy radars and the associated electronic countermeasures (ECM) and electronic counter-countermeasures (ECCM), as well as on ultrawide-band radar. These efforts have involved signal processing, non-Gaussian detection theory, data analysis, system simulation/modeling, threat/ECM modeling, information extraction, and ionospheric/low-altitude/ultrawide-band electromagnetics.

Dr. Mokole is a member of the American Geophysical Union, the American Mathematical Society, the American Physical Society, IEEE Societies on Aerospace and Electronic Systems, Antennas and Propagation, and Geoscience and Remote Sensing, Sigma Xi, and the Society for Industrial and Applied Mathematics.