Computed Fields Near the Edge of a Dielectric Wedge

EGON MARX

Abstract—The behavior of the electromagnetic field near the edge of an infinite sharp dielectric wedge has not been unequivocally established in the theory, and here a numerical experiment is performed to learn more about this behavior. The fields scattered by a finite wedge are determined by solving an integral equation for a function defined on the boundary. The fields near the edge of the wedge are computed from this function by integration. The accepted theory of the fields near the edge of the dielectric wedge is based on a power series expansion that does not exist. The conclusion from this numerical experiment is that only the radial fields along the bisector of the wedge in the transverse magnetic (TM) mode follow the expected power law. The corresponding problem for the perfectly conducting wedge is well understood and is used here to verify the numerical methods.

I. INTRODUCTION

The problem of the scattering of a plane monochromatic wave by an infinite homogeneous dielectric wedge has received much attention mainly because the behavior of the fields near a sharp edge in a cylinder of finite cross section is expected to be that of the fields near the edge of the wedge. Meixner [1] used a power series expansion for the solution of the problem of the infinite wedge and derived the behavior of the fields near the edge from the lowest-order terms. Bach Andersen and Solodukhov [2] showed that terms in the power series in [1] probably diverge if the wedge angle is a rational multiple of π, but they stated that the behavior of the fields should be that of the static fields as found by Meixner. In the same paper they show results of numerical calculations that disagree with the power law they derive. Makarov and Osipov [3] have modified the Meixner series to include powers of log ρ, leaving the lowest-order term unchanged; their approach leads to solutions whose form depends on the angle of the wedge. The behavior of the electromagnetic fields near the edge of a perfectly conducting wedge is well understood [4], [5]: the longitudinal components behave like ρ−t, with t(ρ) between zero and one, and the transverse components diverge like ρ−1. The behavior of static fields in the presence of a dielectric wedge in a dielectric medium is similar.

In this paper we report the results of a numerical experiment in which we compute the fields scattered by a finite wedge terminated by a circular cylinder. To simplify the equations and the computation, we restrict ourselves to the transverse electric (TE) and the transverse magnetic (TM) modes. We reduce the problem to an integral equation for one boundary function. We make the wedge finite to avoid problems with the asymptotic behavior of the scattered field far from the edge. We choose the z-axis along the edge of the wedge, with the wedge located symmetrically about the y-axis below the xz-plane. When the incident wave is a plane monochromatic wave and the geometry is invariant under translations along the z-axis, the electromagnetic fields can be written in terms of two scalar variables, E, and H, and the Maxwell equations reduce to the Helmholtz equation in the xy-plane.

We use the perfectly conducting wedge as a test case because we know the behavior of the fields near the edge and because the resulting integral equations are simpler. In this manner we verify that the numerical methods we use are able to represent correctly the behavior of fields near a sharp edge.

In Section II we recall the main features of the theory of the infinite wedge problem, in Section III we show the equations that we use for the finite wedge, in Section IV we discuss the implementation of the calculations, and in Sections V and VI we show the results for the perfectly conducting wedge and dielectric wedge, respectively.

II. ANALYSIS OF THE INFINITE WEDGE PROBLEM

We designate the region in the xy-plane occupied by the section of the wedge by V and the surrounding region by V, separated by a curve C with a normal n pointing out of the wedge. The field H (E) vanishes in the TE (TM) mode. The longitudinal field E (H) obeys the Helmholtz equation

\[(\nabla^2 + k^2)U(x, y) = 0,\]

where \(k^2 = \varepsilon \omega^2\). The magnetic field in the TE mode is \(H = (i/\mu \omega)E \times \nabla E\) and the electric field in the TM mode is \(E = -(i/\varepsilon \omega)E \times \nabla H\). For static fields we set \(\omega = 0\) in (1) and obtain the Laplace equation.

The solution for the field in the presence of an infinite perfectly conducting wedge of angle ρ is obtained by separation of variables in polar coordinates. The angular function is \(\sin(t \theta)\) and the field vanishes on the sides of the wedge, for \(\theta = 0\) and \(\theta = \pi\), whence the smallest \(t\) is

\[t = \pi/(2\pi - \rho).\]

Since \(0 < \rho < \pi\), we have \(1/2 < t < 1\), and the transverse field diverges like \(\rho^{-1}\).

For static fields in the presence of an infinite dielectric wedge [1], [2], there are two solutions of the Laplace equation that have to be matched at the boundaries. Separation of
variables leads to fields of the form
\[ U(\rho, \theta) = \rho^1 [A(\rho) \sin(\theta) + B(\rho) \cos(\theta)], \]
where \( A \) and \( B \) are power series in \( \rho \). The tangential electric and magnetic fields have to be continuous for \( \theta = 0 \) and \( \theta = \pi \) for all \( \rho \). Consequently, \( E_z \) and \( (1/\mu)\partial E_z/\partial \theta \) or \( H_z \) and \( (1/\epsilon)\partial H_z/\partial \theta \) have to be continuous for these values. The four boundary conditions lead to four homogeneous equations, which have a nontrivial solution only if
\[ D(t) = \sin^2(t \pi) - \rho^2 \sin^2(t(\pi - \nu)) = 0 \]
where \( t = (\mu_2 - \mu_1)/(\mu_2 + \mu_1) \) in the TM mode and \( t = (\epsilon_2 - \epsilon_1)/(\epsilon_2 + \epsilon_1) \) in the TM mode. For each root \( t \) of (4) there is a solution for the static field of the form \( \rho^{1-t} \) times a trigonometric function of \( \theta \), not a power series in \( \rho \). When \( k \) does not vanish, the coefficients in the power series are coupled, and the coefficients of \( \rho^{1-t} \) are determined by inhomogeneous linear equations. The determinant in the denominator of the solutions is now \( D(t + i) \) and \( D(t + i) = D(t) = 0 \) for some \( i \) if \( \nu \) is a rational multiple of \( \pi \). This makes the coefficient infinite unless the numerator also vanishes, which is generally not the case. Thus the power series does not exist for such values of \( \nu \), which form a dense set on the segment of the real line. In [2] the authors assert without proof that the fields near the edge still have the behavior of the static fields.

To summarize, what is expected for the dielectric wedge is that the longitudinal fields will be constant near the edge, corresponding to \( t = 0 \), and that the transverse fields will behave like \( \rho^{1-t} \) for the smallest nonvanishing value of \( t \). For the TE mode, the permeabilities are generally all equal to \( \mu_0 \), whereas \( t = 0 \), and the transverse field would be expected to be constant near the edge due to the term with \( t = 1 \).

III. THE INTEGRAL EQUATIONS

The derivation of the equations that determine the fields scattered by a dielectric cylinder is shown in [6] and [7]. Here we mention only the main results.

If \( U \) satisfies the Helmholtz equation (1), satisfies the radiation condition at infinity, has a jump \( \phi \) across a curve \( C \), and \( \partial U/\partial n \) has a jump \( \eta \) across \( C \), we can write \( U \) in terms of the Green function, which is a Hankel function, and obtain
\[ U = G(\eta) + N(\phi); \]
the functionals \( G(\eta) \) and \( N(\phi) \) are given by
\[ G(\eta)(\vec{x}) = -i(4/\pi) \int_C ds' \eta(s') \tilde{H}_0^1(kR), \]
\[ N(\phi)(\vec{x}) = i(4/\pi) \int_C ds' \phi(s') \tilde{H}_0^1(kR)k\hat{n}' \cdot \vec{R}/R, \]
where \( \vec{x}(s') \) is on \( C \), \( \vec{R} = \vec{x} - \vec{x}', R = |\vec{R}| \), and \( \hat{n}' \) is the unit normal to \( C \) at \( \vec{x}' \). In \( V_1 \) we separate the field into incident and scattered parts, \( E_z = E^i_z + E^s_z \) or \( H_z = H^i_z + H^s_z \), and the scattered part satisfies the radiation condition.

We first solve the problem of the perfectly conducting finite wedge. In the TE mode, we set \( \phi = 0 \) and \( E^s_z(\vec{x}) = G(\eta) \), where a subindex such as 1 on a functional refers to the constants of the medium in \( V_1 \). The field in \( V_2 \) vanishes and we derive the integral equation of the first kind satisfied by \( \eta \),
\[ G(\eta) = -E^o_z, \quad \text{on } C. \]
Alternatively, we set \( \eta = 0 \) and \( E^o_z(x) = N(\phi) \) and derive the integral equation of the second kind,
\[ \phi/2 + N(\phi) = -E^o_z, \quad \text{on } C. \]
In the TM mode, we set \( H^o_z = G(\eta) \) and derive a slightly different integral equation of the second kind,
\[ \eta/2 + N'(\eta) = -\partial H^o_z/\partial n, \quad \text{on } C, \]
where
\[ N'(\eta)(\vec{x}) = \frac{i}{4} \int_C ds' \eta(s') \tilde{H}_1^1(kR)\hat{n}' \cdot \vec{R}/R. \]
For the dielectric cylinder in the TE mode, we set \( \phi = 0 \) and \( E^s_z(\vec{x}) = G(\eta) \). We express the field in \( V_2 \) in terms of \( \eta \) using the boundary conditions and find the integral equation of the first kind,
\[ \frac{1}{2}G(\eta) + \frac{1}{2}aG_2(\eta) + \alpha G_2(\eta) = G_2(\eta) + N_2(\eta) \]
\[ = -\frac{1}{2}E^o_z - \alpha G_2(\eta) \partial E^o_z/\partial n - N_2(E^o_z), \]
where \( \alpha = \mu_2/\mu_1 \). When the argument of one of these functionals is known on the curve \( C \), we can determine \( G \) and \( N \) for all \( \vec{x} \) in the plane, and \( N' \) for all \( \vec{x} \) on \( C \). For the TM mode we obtain the same integral equation (12) with \( H^o_z \) instead of \( E^o_z \) and where now \( \alpha = \epsilon_2/\epsilon_1 \).

IV. NUMERICAL CONSIDERATIONS

To solve the integral equation (8), (9), (10), or (12) we use the point-matching method, which results in a set of linear algebraic equations. The problem with the definition of \( \hat{n} \) at a sharp edge is avoided if no patch is centered there. We also test the effects of using a small circular cylindrical surface matched to the sides of the wedge to approximate the edge, which makes the curvature large but not infinite. The distance from a field point to the edge of a rounded wedge is somewhat arbitrary; we measure this distance along the straight line from that point to the center of the circle.

When the unknown boundary function \( \eta \) is the jump across \( C \) of the normal derivative of a field, we expect \( \eta \) to diverge at the edge, and previous calculations indicate that it does [6], [7]. If we knew the behavior of the boundary function at an edge, we could factor it into the computations and get the correct behavior for the fields.

We specify parameters such as the dielectric constants of the media, the angle \( \nu \) of the wedge, its length \( a \), the angle of incidence \( \theta_0 \), the edge radius \( r \), the location of the field points near the edge on a line at an angle \( \gamma \) with the \( x \)-axis, the wavelength \( \lambda \), the number of patches on the side of the wedge, and the size of the first patch if the distribution of patches is not uniform. The cylindrical surfaces are tangent to the sides of the wedge.
We use a variable patch size to make them about $\lambda/10000$ near the edge and $\lambda/6$ on the large cylinder. These conditions impose a limitation on the size of the wedge $\alpha$, which should be large compared to the wavelength to simulate the infinite wedge. We have taken $\alpha$ comparable to $\lambda$, and we verified that our results do not depend significantly on the choice of $\alpha$. We also have to avoid problems with oscillations in the incident field and in the Hankel functions. We verified that we had a sufficient number of points in the discretization by reducing this number by one-half and checking that the changes in the results were small. The program for the dielectric wedge in the TE mode was verified in the limit $\alpha = 0$, in which the rounded wedge reduces to a circular cylinder. The far fields were compared to those obtained from two programs written for circular cylinders [7]. The changes required to solve the TM problems are minimal. Sections of the code were validated by using them in the programs for the perfectly conducting wedge.

V. RESULTS FOR THE PERFECTLY CONDUCTING WEDGE

We use three different programs, based on (8), (9), and (10), to verify that the computed fields have the correct behavior when we use the different kinds of integral equations. We only show a few representative graphs but draw conclusions from all results.

Polar graphs of the far fields obtained with (8) and (9) for the TE mode are indistinguishable. Also the far fields are not visibly affected by the introduction of the small radius of curvature to replace the sharp edge.

The computed total longitudinal fields near the edge of the wedge are proportional to $\rho^t$, with $t$ given by (2), regardless of the direction $\gamma$ along which the fields are determined. In Fig. 1 we show the longitudinal electric field near the edge calculated via (9) for a sharp wedge and for two rounded ones; the slope in the logarithmic plots is $t$. Results are shown for $\nu = 90^\circ$, and the conclusions remain valid for other angles. The fields calculated via (8) are in even better agreement with the expected behavior for the sharp edge, although the uncertainty in the definition of $\rho$ for a rounded edge remains. The boundary function is a jump in the field or in the normal derivative of a field that is equal to a physical field only in one region, and its behavior is not obvious from the equations. In Fig. 2 we see that the boundary function $\eta$ appears to diverge without following a $\rho^{t-1}$ law. This function becomes more or less constant where the edge is rounded. The boundary function $\phi$ used in the integral equation of the second kind for the sharp wedge does not decrease like $\rho^t$ but is essentially constant for small $\rho$. When the direction of incidence or the direction of the field points bisects the wedge, some terms drop out due to symmetry, and numerical calculations verify these predictions.

VI. RESULTS FOR THE DIELECTRIC WEDGE

The longitudinal fields computed near the edge of a dielectric wedge via (12) are constant, while the transverse fields generally do not show the expected behavior obtained from (4). In Fig. 3 we show the magnitude of the transverse field, $E_x$ or $E_y$, as a function of $\rho$ along the direction of incidence that bisects the wedge for four different values of the dielectric constant. In Fig. 4 we show the boundary functions near the edge for $\varepsilon_2 = 10$ for a sharp wedge and for two rounded wedges. The transverse field near the edge appears to diverge, but the agreement with the expected behavior is poor compared to that obtained for the perfect conductors. Similar curves are obtained when the direction of incidence, the polar angle of the field points, or the wedge angle is changed. The boundary function shows divergent behavior near a sharp edge, and for a finite radius the boundary function tends to be
MAW: COMPUTED FIELDS NEAR EDGE OF DIELECTRIC WEDGE

Fig. 4. Boundary functions for different edge radii, TM mode.

Fig. 5. Radial component of the field along different directions, TM mode.

Fig. 6. Radial component of the field along different directions, TE mode.

constant over the circular part, as was the case for the perfect conductor. For a sharp wedge the first two points tend to line up with slope $r^{-1}$, although the alignment shown in Fig. 4 is unusually good. The spikes in the boundary function for a wedge of radius $r$ at a distance $r
\nu/2$ from the vertex correspond to the points where the cylinder is matched to the plane, that is, where the curvature is discontinuous. The agreement between the three curves is good on the side of the wedge. The far fields are not significantly affected by the singular behavior of the boundary function near the edge. The only evidence of the expected behavior that we found took place along the bisector, as seen for $H_\nu$ in the TM mode in Fig. 5 and $E_\nu$ in the TE mode in Fig. 6. Since the incident field is bounded, only the scattered field can diverge. The radial component is proportional to

$$F^s_\rho(x) = \int \frac{ds' \eta(s') H_\nu^{(1)}(kR)(x - x') \sin \gamma - (y - y') \cos \gamma}{R} \, \frac{1}{R}, \quad (13)$$

which shows the special symmetry when the field point is on the $y$-axis, where $\gamma = \pi/2$ and $x = 0$. If the incidence is also along the bisector, the radial component of the field vanishes.

The difficulty of finding an analytic solution for the scattering of a plane monochromatic wave by an infinite dielectric wedge problem may be due to the fact that we have to match two fields that propagate with different constants along two boundaries that are not parallel. The existence of such a solution to the scattering problem remains open to question. This is not the case for the perfect conductor and for the static fields. To replace a sharp edge with a rounded edge may help in computations, but this method does not work where three regions meet because rounding off one region produces a sharp spike in another.

We have computed the fields in the TE and TM modes. A wave incident on a dielectric wedge with arbitrary polarization and direction of incidence gives rise to a system of two integral equations for two boundary functions $\eta$, $\eta'$, and we expect the results to be similar.

VII. CONCLUSION

The behavior of the computed fields near the edge of a dielectric wedge disagrees with the power law derived for static fields. The exception occurs for the radial components along the bisector of the wedge, a configuration that has a special symmetry. Thus the results indicate that there may be a somewhat more singular term in the expansion that vanishes under these special conditions.

The singularity of the boundary functions near the edge of the wedge may cause numerical difficulties unless we know the nature of the divergence and can factor it into the computer programs. Further efforts are required to find a rigorous solution to the fields near the edge of a dielectric wedge, possibly along the lines proposed in [3]. The behavior of the boundary function does not follow directly from the behavior of the fields, as seen even for perfect conductors.

The method of solution for problems of scattering by dielectric obstacles via integral equations can be used for scatterers...
with sharp edges with caution. The far fields and radar cross sections do not depend strongly on the treatment of sharp edges, but the computed near fields do.

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REFERENCES

Egon Marx, for a photograph and biography please see page 628 of the May 1989 issue of this TRANSACTIONS.