


Computation of Monopole Antenna Currents Using Cylindrical Harmonics

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Abstract—A novel method for computing the input impedance and induced currents on cylindrical antenna structures is investigated. This technique considers the antenna when placed between two parallel ground planes, thus allowing the use of cylindrical harmonic Fourier expansions to represent the fields. Enforcement of boundary and continuity conditions on the tangential field component results in solutions for the discrete spectral coefficients of these expansions. Convergence is discussed and computational validations are presented for varied cases.

INTRODUCTION

The computation of induced currents on conducting antenna structures is usually performed using a discretized integral equation, as per the moment method [1]. An alternate procedure is to use either a numerical formulation based upon finite elements or finite differences [2]. A third option is investigated here, whereby separable coordinate solutions to Maxwell’s equations in multiple regions are employed for the simple case of a thin-wire monopole of finite length [3].

An eigenfunction expansion is used for evaluating the induced current on a simple monopole antenna radiating into the half-space over a perfect electric conducting (PEC) ground plane. To apply the method, the monopole is enclosed by adding a second, parallel ground plane, as is illustrated in Fig. 1. The original unbounded half-space region is therefore replaced with an infinite sequence of images. This approximation allows the fields to be represented using cylindrical harmonics in each of the two subregions between the ground planes. Coefficients of these expansions are found by enforcing appropriate boundary or continuity conditions on the tangential field components. A similar approach has been used to compute currents on a top-hat monopole [4]. After evaluating $I(z)$, which is our concern here, the radiated fields can be found using a free-space Green’s function integration over the monopole without consideration of the upper ground plane.

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A circular cylindrical coordinate system $(\rho, \phi, z)$ is oriented so that the bottom ground plane in Fig. 1 lies in the $x$-$y$ plane and the $z$-axis is vertically centered through the monopole. An electric type (TM) Hertz potential is used to represent the field components [5, ch. 5]. Assuming axisymmetric fields, with no $\phi$-variation, the Helmholtz equation for the Hertz potential is easily solved using cylindrical harmonics in each of the two regions. Enforcement of the PEC conditions ($E_z = 0$) on both ground planes and on the top of the monopole yields eigenmodes corresponding to Fourier series in the $z$-coordinate for the periodic, infinitely imaged structure. Truncated expansions for nonzero field components in the spatial regions of Fig. 1 result from this development.

In Region I ($\rho \geq a$ and $0 \leq z \leq l$), we have

$$E_{1\parallel}(\rho, z) = \frac{1}{\hat{y}} \sum_{n=0}^{N_1} a_n v_n H_0^{(1)}(v_n \rho) \cos \left( \frac{n\pi z}{l} \right)$$

$$H_{1\parallel}(\rho, z) = \sum_{n=0}^{N_1} a_n v_n H_1^{(1)}(v_n \rho) \cos \left( \frac{n\pi z}{l} \right)$$

with $v_n = \sqrt{k_0^2 - (n\pi/l)^2}$ and $\hat{y} = j\omega_0$.

In Region II ($\rho \leq a$ and $h \leq z \leq l$), there results

$$E_{2\parallel}(\rho, z) = \frac{1}{\hat{y}} \sum_{n=0}^{N_2} c_n u_n J_1(u_n \rho) \cos \left( \frac{n\pi z}{q} (z - h) \right)$$

$$H_{2\parallel}(\rho, z) = \sum_{n=0}^{N_2} c_n u_n J_0(u_n \rho) \cos \left( \frac{n\pi z}{q} (z - h) \right)$$

with $u_n = \sqrt{k_0^2 - (n\pi/q)^2}$ and $q = l - h$. The $J_m$ and $H_m^{(2)}$ are, respectively, Bessel function of the first kind and Hankel function of the second kind, of order $m = 0$ or 1.

The expansion coefficients are found by least-squares enforcement of the remaining boundary and continuity conditions on the tangential fields. Among these constraints is the PEC condition on the wire surface, augmented by an assumed uniform $E_z$-field in the gap having voltage $V_0$,

$$E_{2\parallel}(\rho, z) = \begin{cases} -V_0/d, & \text{for } 0 \leq z \leq d \\ 0, & \text{for } d \leq z \leq h \end{cases}$$

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Additionally, we least-squares match the corresponding Region I and Region II expansions for $E_r(a, z)$ and $H_r(a, z)$ along the interface at $\rho = d$ and $h < z < l$. Since the expansions in (1) and (2) are Fourier series, using orthogonal basis functions in the $z$-coordinate, the least-squares equalities generate simple moment integrations, defined by

$$I_m^{(1)} = \int_0^d \cos \left( \frac{m \pi}{l} z \right) dz = \left\{ \begin{array}{ll} l, & \text{for } m = 0 \\ l/2, & \text{for } m \neq 0 \end{array} \right. \quad (4a)$$

$$I_m^{(2)} = \int_0^d \cos \left( \frac{m \pi}{q} z \right) dz = \left\{ \begin{array}{ll} q, & \text{for } m = 0 \\ q/2, & \text{for } m \neq 0 \end{array} \right. \quad (4b)$$

$$I_m^{(3)} = \int_0^d \cos \left( \frac{m \pi}{l} z \right) dz = \left\{ \begin{array}{ll} d, & \text{for } m = 0 \\ \frac{l}{m \pi} \sin \left( \frac{m \pi d}{l} \right), & \text{for } m \neq 0 \end{array} \right. \quad (4c)$$

The $c_m$ unknowns can be eliminated by a judicious substitution within the Fourier moment equations, resulting in the linear system

$$\sum_{k=0}^{N_1} A_{m,k} a_k = B_m, \quad \text{for } m = 0, 1, \ldots, N_1 \quad (5a)$$

where

$$A_{m,k} = r_k H_1^{(2)}(r_k a) \sum_{n=0}^{N_1} \frac{u_n J_0(u_n a) T_{m,n} T_k}{J_1(u_n a)} I_n^{(2)} - r_k H_0^{(2)}(r_k a) I_m^{(1)} b_{m,k} \quad (5b)$$

and

$$B_m = \frac{y}{d} I_m^{(1)} \quad (5c)$$

with the Kronecker delta in (5b) defined by $b_{m,k} = 1$ for $m = k$ and zero otherwise.

Upon inverting the linear system in (5a), the monopole currents are found by using

$$I(z) = 2\pi a I_0^{(2)}(a, z) = 2\pi a \sum_{n=0}^{N_1} \frac{a_n r_n H_1^{(2)}(r_n a) \cos \left( \frac{n \pi z}{l} \right)}{l} \quad (6)$$

It is important to note that there is no explicit enforcement of $I(z) \to 0$ at the top of the monopole, as is the case in most integral equation formulations. As it turns out, this end-condition will be satisfied naturally as a result of enforcing the necessary boundary and continuity conditions on the tangential field components.

As an optional final step, the radiated field can be evaluated, for instance, by using a far-field approximation to the free-space Green's function integration over the dipole composed of the monopole and its image. [5, sec. 3-13]. Note that the presence of the upper ground plane is ignored in this final step—the monopole antenna is considered to be radiating into an unbounded half-space.

**VALIDATION**

Extensive checks were performed using the Numerical Electromagnetics Code (NEC) [6] as a benchmark for computed currents.

In using the NEC, the effect of the upper ground plane was approximated by constructing a finite number of periodically spaced dipole images. The required number of images used in the NEC calculations was determined by numerical experiment. For the cases considered here, it was found that a single pair of dipole images placed at $z = \pm 2l$ was sufficient for convergence to $\leq 1\%$. This result is illustrated in Fig. 2, where a monopole having $h = 0.24 \lambda_0$ and $d = 0.01 \lambda_0$ is being considered. The NEC computed $|I(z)|$ is shown for three cases: no upper ground plane (denoted by zero images), one image set and two image sets. Geometrical parameters: $h = 0.24 \lambda_0$, $a = 0.01 \lambda_0$, and $l = 1.4 \lambda_0$.

The validations of our computer program, "MONO," included tests for series truncation convergence, via adjustment of $N_1$ and $N_2$ as functions of the following: monopole height $h$, ground plane spacing $l$, wire radius $a$, and gap spacing $d$. Results showed that, as a rule-of-thumb, about 20 modes ($N_1$ or $N_2$) per $\lambda_0$ of $z$-axis distance are needed for convergence of the $I(z)$ solution on the monopole to within $2\%$.

Figs. 3–5 display some example validations for MONO. Series truncation convergence is illustrated in Fig. 3, where the $|I(z)|$ for $N_1 = N_2 = 60$ is compared to that for $N_1 = N_2 = 100$, with monopole height $h = 0.24 \lambda_0$ and a ground plane spacing of $l = 3.1 \lambda_0$. The phase of $I(z)$ showed comparable convergence to that of the displayed magnitude.

The effect of ground plane spacing was investigated by comparing MONO and NEC computations for three cases: $l = 3.1$, 1.40, and 0.81 $\lambda_0$. The largest deviation was observed for $l = 0.81 \lambda_0$ spacing;
The rationale behind this partitioning was the expectation that a highly accurate approximation to the $E_i^I(a, z)$, as given by (3), would be required in order to attain an accurate solution for $I(z)$. Due to the narrow gap $d$, a very large $N\!_{1}$ is needed, typically measured in the hundreds. Requisite inversion of the $N\!_{1} \times N\!_{1}$ $A$-matrix, as given by (5), would result in an inefficient and virtually unusable numerical algorithm. To circumvent the generation of this unwieldy matrix, while retaining an accurate $E_i^I(a, z)$, led to the dual summation approach in (7). The idea was to use superposition, with

$$E_i^I(a, z) = E_i^{I_1}(a, z) + E_i^{I_2}(a, z) = \sum_{n=0}^{N_1} \sum_{\nu=0}^{N_2} a_{\nu} H_i^{(2)}(\nu \rho, \rho) \cos \left( \frac{n \pi z}{I} \right)$$

(7a)

and

$$H_i^{(2)}(\rho, z) = H_i^{(1)}(\rho, z) + H_i^{(2)}(\rho, z) = \sum_{n=0}^{N_1} b_{n} \rho H_i^{(1)}(\nu \rho, \rho) \cos \left( \frac{n \pi z}{I} \right)$$

(7b)

The purpose of the $b_{n}$ expansion in (7) is to enforce the narrow gap condition, as per (3). The $b_{n}$ can be computed directly through the least-squares enforcement of (8b); these coefficients are then used to drive the remaining equations for the unknown $a_{\nu}$ and $c_{\nu}$. The upper index $M_1$, in the $E_i^I(a, z)$ summation should satisfy $M_1 \geq 1/d$ in order to approximate the hypothetical step function behavior across the narrow gap. As can be seen in Fig. 6, for the case of $M_1 = 3/d = 477$, there is elusive pointwise convergence near the feed gap, as is well known from the "Gibb's phenomenon" [7].

In performing numerical experiments, whereby the value of $M_1$ was varied, it was found that the convergence of $I(z)$ was much faster than that for $E_i^I(a, z)$. This is depicted in Figs. 7 and 8 for

**TABLE I RESULTS FOR VARIOUS GROUND PLANE SPACINGS WHEN $h = 0.24 \lambda_0$, $a = 0.01 \lambda_0$, AND $d = 0.05 \lambda_0$**

<table>
<thead>
<tr>
<th>Plate Spacing</th>
<th>NEC</th>
<th>MONO</th>
<th>% Difference</th>
<th>$R$</th>
<th>$X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.10 $\lambda$</td>
<td>44.63</td>
<td>33.35</td>
<td>42.65</td>
<td>13.14</td>
<td>4.64</td>
</tr>
<tr>
<td>1.40 $\lambda$</td>
<td>44.90</td>
<td>13.97</td>
<td>43.14</td>
<td>13.98</td>
<td>4.08</td>
</tr>
<tr>
<td>0.81 $\lambda$</td>
<td>43.51</td>
<td>14.84</td>
<td>44.67</td>
<td>13.84</td>
<td>2.66</td>
</tr>
</tbody>
</table>

this case is illustrated in Fig. 4. It should be noted that the upper ground plane is only 0.57 $\lambda_0$ above the top of the monopole for this calculation.

A summary of the results for the three ground plane spacings is given in Table I. The best agreement is seen to occur for $l = 1.40 \lambda_0$. This anomaly may be due to the use of a fixed number of modes $N_1 = N_2 = 60$ in all three cases, thus yielding a little better convergence for the 1.40 $\lambda_0$ case vis-a-vis that for the 3.10 $\lambda_0$ spacing.

A final comparison of NEC and MONO is shown in Fig. 5, where the case of an $h = 0.48 \lambda_0$ monopole is considered.

**DISCUSSION**

In the initial stages of formulation development, the Region I expansions in (1) were each decomposed into two separate series of the form

$$E_i^I(\rho, z) = E_i^{I_1}(\rho, z) + E_i^{I_2}(\rho, z)$$

$$= \sum_{\nu=0}^{N_1} a_{\nu} H_i^{(2)}(\nu \rho, \rho) \cos \left( \frac{n \pi z}{I} \right)$$

(7a)

and

$$H_i^{(2)}(\rho, z) = H_i^{(1)}(\rho, z) + H_i^{(2)}(\rho, z)$$

(7b)

Due to the "Gibb's phenomenon" [7].

The purpose of the $b_{n}$ expansion in (7) is to enforce the narrow gap condition, as per (3). The $b_{n}$ can be computed directly through the least-squares enforcement of (8b); these coefficients are then used to drive the remaining equations for the unknown $a_{\nu}$ and $c_{\nu}$. The upper index $M_1$, in the $E_i^I(a, z)$ summation should satisfy $M_1 \geq 1/d$ in order to approximate the hypothetical step function behavior across the narrow gap. As can be seen in Fig. 6, for the case of $M_1 = 3/d = 477$, there is elusive pointwise convergence near the feed gap, as is well known from the "Gibb's phenomenon" [7].
Fig. 7. Comparison of $E_1(a, z)$ along the monopole for $M_1 = 30$ and $M_1 = 217$ with $h = 0.24 \lambda_0$, $a = 0.01 \lambda_0$, $d = 0.05 \lambda_0$, $l = 1.4 \lambda_0$ and $N_1 = N_2 = 60$.

Fig. 8. Comparison of $I(z)$ along the monopole for $M_1 = 30$ and $M_1 = 217$ with $h = 0.24 \lambda_0$, $a = 0.01 \lambda_0$, $d = 0.05 \lambda_0$, $l = 1.4 \lambda_0$ and $N_1 = N_2 = 60$.

the case of $h = 0.24 \lambda_0$, where the respective $E_1(a, z)$ and $I(z)$ for $M_1 = 30$ are compared along the monopole to those for $M_1 = 217$. As is evident, the narrow pulse function for $E_1(a, z)$ across the gap requires a broad spectrum of Fourier terms for accurate least-squares reconstruction. There was no mandate, however, to have good convergence of $E_1(a, z)$ to assure an accurate $I(z)$. This result led to abandonment of the augmented $b_n$ series, as shown in (7).

A final observation is that the upper ground plane can be brought to within less than $\lambda_0$ of the top of the monopole without any major effect on the computed current distribution vis-a-vis the open half-space case. Special consideration is needed, however, when either $l/d$ is an integer or $q = l - h$ equals an exact multiple of $\lambda_0/2$. These special cases lead, respectively, to a zero of the driver of (5c), through $l_0^{(a)} = 0$, or an indeterminant in (5b), via the malling of one or more of the $u_n$ and, hence, the associated $J_1(u_n \alpha)$.

In summary, the purpose of this work has been to investigate initially the viability of using cylindrical harmonic expansions between parallel plates to approximate the induced currents on radiating and scattering structures in unbounded space. Further extensions to the simple monopole example will involve thin-wire scattering and more complicated antenna structures. A particular case will be the installation of a top-plate with inhomogeneous dielectric loading below the plate to focus the radiation pattern in desired directions.

REFERENCES


Natural Resonances of Conducting Bodies of Revolution

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Abstract—The complex plane natural resonances for several perfectly conducting bodies of revolution (sphere, right circular cylinder, prolate spheroid, and circular disk) are calculated using electric, magnetic, and combined field integral equation formulations (EFIE, HFIE, and CFIE, respectively). These results are compared with some recently published values, and discrepancies are noted and discussed.

I. INTRODUCTION

The determination (analytically and numerically) of the natural resonances of conducting antennas and scatterers has received considerable attention from a number of investigators in the last two decades (Baum [1]; Martinez et al. [2]; Marin [3]; Tesche [4]; Umashankar et al. [5]; Crow et al. [6]; Roberts and Pearson [7]; Kristensson [8]). In a recent issue, Merchant et al. [9] have presented these complex resonances for the sphere, several prolate spheroids, and right circular cylinders. In particular, this work has addressed the patterns of the complex resonances for the important and interesting cases of azimuthally varying modal current and charge distributions (exp $\text{im}\theta$) where $m \neq 0$. While this work of Merchant et al. is most interesting, and for the most part numerically accurate, it has come to our attention in attempting to reproduce these complex resonances that some discrepancies have been found for the resonances associated with current and charge modes which have azimuthal variation. It is the purpose of this communication to address (and hopefully correct) these minor discrepancies for the benefit of future investigators.

All results here follow the usual singularity expansion method (SEM) convention where the complex natural resonances occur in the left half-plane due to the use of a Laplace transform. Some investigators, including [8] and [9], employ a complex Fourier transform which places the singularities in the lower half-plane. Thus Manuscript received April 20, 1989; revised September 21, 1989. This work was supported by the Electromagnetic Signature Consortium (NASA, U.S. Army, U.S. Navy, U.S. Air Force).

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