Selecting a Reduced-Rank Transformation for STAP—A Direct Form Perspective

We introduce the signal-to-interference plus noise ratio (SINR) metric as a tool for selecting a rank reduction transformation that maximizes the output SINR of a direct form space-time adaptive processor. The SINR metric is used to identify the eigenvectors with the greatest impact on the output SINR which become the columns of the rank reduction transformation. The SINR metric method exhibits a graceful degradation in performance as the rank of the transformation approaches one.

I. INTRODUCTION

Recently, Goldstein and Reed [1–4] have proposed a reduced-rank (RR) space-time adaptive processing (STAP) method based on the generalized sidelobe canceler (GSC). This proposed method provides a graceful degradation in performance as the rank of the transformation is reduced below the rank of the interference subspace. In contrast, principal component methods, such as the principal component inverse (PCI) [5] and minimum norm eigencanceler [6], can exhibit a sharp decrease in performance if the number of principal components is underestimated (i.e., the principal components do not span the interference subspace) [1]. Goldstein and Reed introduce the output signal-to-interference plus noise ratio (SINR) as a cost function into the process of selecting the transformation. With the Goldstein and Reed method, the columns of the transformation are selected based on a cross-spectral metric (CSM) as opposed to selecting the eigenvectors associated with the principal components (largest eigenvalues). The CSM is used to identify the eigenvectors which have the greatest impact on the output SINR and in turn, these eigenvectors become the columns of the rank reduction transformation.

We propose an RR method, which we refer to as the SINR metric method, that also uses the output SINR as a cost function and exhibits a graceful degradation in performance as the transformation rank is reduced. In contrast to the Goldstein and Reed GSC method, the proposed method is presented from a direct form perspective. We introduce the SINR metric as the direct form analog to the CSM. The SINR metric is used to identify the eigenvectors of the interference plus noise correlation matrix which minimize the loss in SINR performance and in turn, these eigenvectors become the columns of the transformation.

Although the CSM and SINR metric methods provide a graceful degradation in performance as the rank is reduced, the computational cost of both of these methods (and in general, any eigen-based method) is an issue due to the high computational cost associated with the eigendecomposition. Further, one must consider the cost of performing the rank reduction transformation on the signals, evaluating the cost function, and sorting/selecting the eigenvectors based on the cost function. In the end, these additional computational costs may be counter productive to the goal of reducing the high computational costs associated with STAP.

The rest of this work is organized as follows. The SINR metric method is presented in Section II with simulation results presented in Section III. Section IV provides some concluding remarks.

II. SINR METRIC METHOD

Goldstein and Reed introduced the CSM based on the RR GSC structure shown in Fig. 1. The output SINR from the GSC shown in Fig. 1 (ignoring Z) is [1]

\[
\text{SINR}_{gsc} = \frac{|\alpha|^2}{\sigma_d^2 - r_{bd}^H R_b^{-1} r_{bd}} = \frac{|\alpha|^2}{\sigma_d^2 - \sum_{i=1}^{N} \frac{|u_i^H r_{bd}|^2}{\lambda_i}}
\]

(1)

where \( R_b = BRB^H \), \( B \) is the \( KJ \times KJ \) interference plus noise correlation matrix, \( B \) is the \( KJ - 1 \times KJ \) blocking matrix, \( r_{bd} = BRs \), \( \sigma_d^2 = s^H Rs \), \( s \) is the \( KJ \times 1 \) steering vector, \( |\alpha|^2 \) is the power of the desired signal, \( \{u_i\}_{i=1}^{N} \) are the eigenvectors of \( R_b \), \( \{\lambda_i\}_{i=1}^{N} \) are the associated eigenvalues, \( N = KJ - 1 \), \( K \) is the number of antenna elements, \( J \) is the number of pulses in a coherent processing interval, and \((\cdot)^H\) denotes conjugate transpose. The term

\[
\frac{|u_i^H r_{bd}|^2}{\lambda_i}
\]

(2)
is the CSM [1]. When the rank reduction transformation $Z$ is composed of $r$ distinct eigenvectors of $R_R$, the summation in (1) is only a function of those $r$ eigenvectors in $Z$. Clearly, the SINR is maximized by selecting the $r$ eigenvectors with the largest CSM.

Here we consider the RR direct form processor shown in Fig. 2. The $KJ \times r$ matrix $V$ is the rank reduction transformation and $w$ is an $r \times 1$ weight vector designed to maximize the SINR. The full dimension (i.e., without $V$) weight vector of the direct form processor is given by

$$w = R^{-1}s$$  \hspace{1cm} (3)

and the RR weight vector by

$$w = (V^H RV)^{-1}V^H s$$  \hspace{1cm} (4)

where $R$ is the correlation matrix of the interference plus noise and $s$ is the steering vector. As with the RR GSC, the SINR performance of the RR direct form processor will be less than the full dimension direct form processor. Thus, the objective is to select the $r$ columns of $V$ such that the loss in SINR performance is minimized. Inspired by Goldstein and Reed’s CSM method, we propose a method where the columns of $V$ are eigenvectors of $R$ and are selected based on their impact on the output SINR from a direct form perspective. That is, we are using the output SINR as a cost function to identify the $r$ eigenvectors (columns of $V$) that minimize the loss in SINR performance.

Without the rank reduction transformations, the GSC shown in Fig. 1 provides the same output SINR as the direct form method shown in Fig. 2 when the weight vector $w$ is defined by (3) and $w_s = R^{-1}f_{BR}$ [1]. However, the SINR equations for the full dimension GSC and direct form processor are different. The SINR equation for the full dimension direct form processor is [1]

$$\text{SINR}_{dmi} = |{\alpha}|^2 s^H R^{-1} s = |{\alpha}|^2 \sum_{i=1}^{KJ} \frac{|f_i^H s|^2}{\lambda_i}$$  \hspace{1cm} (5)

where $\{f_i\}_{i=1}^{KJ}$ are the eigenvectors of $R$ and $\{\tilde{\lambda}_i\}_{i=1}^{KJ}$ are the associated eigenvalues. With the RR weight vector defined as in (4), the output SINR for the RR direct form processor is given by [7]

$$\text{SINR}_{RRdmi} = |{\alpha}|^2 s^H V (V^H RV)^{-1} V^H s.$$  \hspace{1cm} (6)

Now, if we restricted the $r$ columns of $V$ be a unique subset of the eigenvectors of $R$, then we can rewrite

\begin{equation}
\text{SINR}_{RRdmi} = |{\alpha}|^2 s^H V \tilde{A}^{-1} V^H s = |{\alpha}|^2 \sum_{i=1}^{r} \frac{|v_i^H s|^2}{\tilde{\lambda}_i}
\end{equation}  \hspace{1cm} (7)

where $\tilde{A}$ is a diagonal matrix of the eigenvalues associated with the $r$ eigenvectors and $\{v_i\}_{i=1}^{r}$ denote the columns of $V$. The output SINR of the RR direct form processor given by (7) is only a partial sum of the output SINR for the fully adaptive processor given by (5). Thus, the RR direct form processor will incur a loss in SINR performance. The objective is to select the columns of $V$ as the eigenvectors of $R$ that minimize the loss in SINR performance, which is equivalent to maximizing the partial sum given in (7). Clearly, the partial sum is maximized by selecting the $r$ columns of $V$ to be the eigenvectors which maximize the quantity

$$\frac{|f_i^H s|^2}{\lambda_i}$$  \hspace{1cm} (8)

which we refer to as the SINR metric.

The eigenvectors $\{f_i\}_{i=1}^{KJ}$ form an orthonormal basis for the signal space and thus, the steering vector and the interference plus noise vector can be written as a linear combination of the eigenvectors. The SINR metric given by (8) basically represents the SINR along each of the basis vectors. We denote the subspace spanned by the $r$ eigenvectors with the highest SINR metric as the high SINR subspace and the subspace spanned by the remaining eigenvectors as the low SINR subspace. Now, note that the high SINR subspace is orthogonal to the low SINR subspace. If we select the $r$ columns of $V$ to be the eigenvectors with the largest SINR metric, then the weight vector defined by (4) lies in the high SINR subspace and thus, cancels any signal components (eigenvectors) in the low SINR subspace.

In summary, both the CSM and SINR metric methods introduce the output SINR as a cost function into the process of selecting the rank reduction transformation. The CSM and SINR metric provide an ordering of the eigenvectors of $BRB^H$ and $R$, respectively, based on their relative impact on the output SINR. In contrast, ordering the eigenvectors according to the principal components (i.e., largest eigenvalues) is not directly related to the output SINR. The principal component (PC) ordering provides the best low rank approximation to the full dimension matrix [8]. One can show that the CSM GSC and PC GSC provide the same SINR performance when the rank of the transformation exceeds the dimension of the interference subspace. However, as Goldstein and Reed have shown, the best low rank approximation does not translate into maximizing the output SINR when the rank is below the dimension of the interference subspace. One can also show that the full dimension direct form processor and GSC
have the same SINR performance, but as the rank is reduced below full dimension, a direct analytical comparison of the CSM and SINR metric methods becomes a difficult task. The SINR performance of the CSM and SINR metric methods as a function of the transformation rank \( r \) (i.e., the number of columns in the rank reduction transformation) is determined by their respective SINR equations:

\[
\text{SINR}_{\text{gsc}}(r) = \frac{|\alpha|^2}{\sigma_d^2 - \sum_{r'}|u^{r'}_d r_{bd}|^2 \lambda_j} \\
\text{for } 1 \leq r \leq KJ - 1
\]

\[
\text{SINR}_{\text{dmi}}(r) = \frac{|\alpha|^2}{\lambda_j \sum_{i=1}^{r} f_i^r s_i^2} \\
\text{for } 1 \leq r \leq KJ
\]

where \( r \) is the number of columns in the rank reduction transformation. Basically, the relative performance of the CSM and SINR metric methods will depend on the rate that (9) and (10) increase as a function of \( r \). The direct relationship between (9) and (10) as well as the behavior of \( u^{r'}_d r_{bd} \) and \( f_i^r s_i \) as a function of the steering vector and the interference plus noise environment remain as open research areas.

In the next section, we present simulation results that show the SINR performance of the CSM GSC and SINR metric direct form processor in several scenarios. Our intent in showing the simulation results is not to suggest that one method is better than the other, but to show the importance of incorporating a cost function in the process of selecting the rank reduction transformation. Additionally, the simulation results highlight that the SINR performance of each method is dependent on the interference plus noise environment and the available resources (i.e., transformation rank).

### III. SIMULATION RESULTS

In this section, we examine the SINR performance as a function of the number of eigenvectors used in the rank reduction transformation for the SINR metric method, the CSM GSC method, and a PC version of the GSC processor. With the PC GSC, the columns of \( Z \) are filled with the eigenvectors associated with the largest eigenvalues of \( BRB^T \). We also present simulation results from a hypothetical direct form processor where the columns of \( V \) are filled with the eigenvectors associated with the smallest eigenvalues, which we refer to as the small method. Recall that each eigenvalue gives an indication of the interference plus noise power level along its associated eigenvector. The small method represents the heuristic reasoning that one should select the eigenvectors with the smallest eigenvalues because the interference plus noise power level along these eigenvectors is the smallest. Basically, the small method forces the weight vector \( w \) to lie in a subspace orthogonal to the eigenvectors with higher interference plus noise power levels and thus, leading to their cancellation.

The simulated radar had a linear array of 14 antenna elements spaced at half a wavelength with 16 pulses in a coherent processing interval. The interference environment consisted of three barrage noise jammers and clutter. The three jammers were at normalized angles of 0.25, 0.433, and \(-0.433\), and are referred to as Jammer 1, Jammer 2, and Jammer 3, respectively.

The clutter was simulated by 360 point scatters evenly distributed in azimuth and with the radar parameters selected such that \( \beta = 1 \), where \( \beta \) defines the slope of the clutter ridge across the normalized Doppler/angular plane (see [9, pp. 24–28] for a complete discussion of \( \beta \)). The simulation results from six cases (scenarios) are presented with the relevant parameters listed in Table I. Cases 1–3 represent a high clutter and jamming environment and Cases 4–6 represent a low clutter and jamming environment.

Figs. 3 and 4 show the SINR performance of the four methods as a function of the number of columns (transformation rank) in the rank reduction transformation for Cases 1 and 4, respectively. Plotted in Figs. 3 and 4 are (9) with the summation ordered by the largest CSM and by the largest (PC) eigenvalues and (10) with the summation ordered by the largest SINR metric and the smallest eigenvalues. An examination of the plots in Figs. 3 and 4 reveals that the SINR metric method outperforms the small
method and the CSM GSC method outperforms the PC GSC method as the transformation rank is reduced below full dimension, attesting to the importance of incorporating a cost function into the process of selecting the rank reduction transformation. Similar plots for the other cases show the same characteristics in regards to the difference in SINR performance of the SINR metric method and CSM GSC method with their respective counterparts. Note that additional comments on the small method are provided at the end of this section.

A further examination of Figs. 3 and 4 reveals that above a certain rank the CSM GSC method outperforms the SINR metric method and below this rank, the converse is true. As shown in Figs. 5 and 6, the SINR performance curve crossover trend holds for all the cases in Table I. Fig. 5 shows the performance curves (SINR versus transformation rank) for SINR metric and CSM GSC methods in the high clutter and jamming environment (Case 1–3). Fig. 6 shows the performance curves for the low clutter and jamming environment (Cases 4–6). Observe from Figs. 5 and 6 that the crossover rank for the SINR performance curves for the SINR metric and CSM GSC methods varies as the interference plus noise environment and steering vector change. At this time, the determination of the crossover rank remains an open research area and its resolution will depend on resolving the relationship between (9) and (10). In general, we expect the crossover rank to move to the left as the dimension of interference subspace decreases, since the RR GSC achieves optimal SINR performance when the rank of the transformation matches the dimension of the interference subspace. However, by the same token, we expect the crossover rank to move to the right as the dimension of the interference subspace increases. Figs. 5 and 6 highlight that the SINR performance of the two given methods, and
most likely any STAP method, is dependent on the scenario as well as the transformation rank. Thus, one cannot in general claim that one method is “better” than another without specifying the scenario and the transformation rank. We now examine Cases 1 and 4 in more detail to gain additional insight into the difference between the SINR metric and CSM GSC methods, starting with Case 1.

First, observe that the interference plus noise correlation matrix has a rank of approximately 77 and as noted previously, the PC method displays a sharp decrease in performance as the transformation rank decreases below 77 (see Fig. 3). Both the CSM and SINR metric methods display a more graceful degradation in performance as the transformation rank decreased below 77. The SINR metric method has better performance as the transformation is decreased below 57. The eigenvalues of $R_b$ and the SINR metric for Case 1 are plotted in Fig. 7. Fig. 7 reveals that the largest SINR metrics occur in the noise subspace (i.e., the eigenvectors associated with the smallest eigenvalues) which is orthogonal to the interference subspace. Thus, even as the rank of the transformation approaches 1, the SINR metric method will provide a weight vector that lies in a subspace orthogonal to the interference subspace which effectively cancels all the interference. The SINR performance of CSM and and PC GSC is significantly less than the SINR metric method at a transformation rank of 1 because the GSC can only cancel a single interference component leaving approximately 76 interference components un-cancelled. The eigenvalues of $R_b$ and the CSM are plotted in Fig. 8. Fig. 8 shows that the largest eigenvalues do not necessarily correspond to the largest CSM. The SINR performance of the PC GSC is less than the CSM GSC because the sum in (9) will not be maximized when the eigenvectors associated

Fig. 5. SINR performance of SINR metric and CSM GSC methods as function of transformation rank for high clutter and jamming environment (Cases 1–3).

Fig. 6. SINR performance of SINR metric and CSM GSC methods as function of transformation rank for low clutter and jamming environment (Cases 4–6).
with largest eigenvalues are used. The interference plus noise power after upper branch processing, $\sigma_d^2$, is also plotted in Fig. 8.

The difference between Case 4 and Case 1 is the reduction of the clutter-to-noise ratio from 55 dB to 20 dB and the jammer-to-noise ratio for each jammer 0 dB. A comparison of Figs. 3 and 4 reveals that the SINR metric method has the same basic performance in both cases, while the performance of the CSM and PC GSC is better in Case 4 than in Case 1. An examination of the SINR and CSM metric plots for Case 4 (see Figs. 9 and 10) provides insight into explaining these observations. As Fig. 9 shows, a large percentage of the highest SINR metrics occur in the noise subspace and in fact, the largest SINR metric occurs in the noise subspace. Thus, as the transformation rank approaches 1, the weight vector $\mathbf{w}$ lies in a subspace orthogonal to the interference subspace providing complete cancellation of the interference. Since the power in the noise subspace did not change from Case 1 to Case 4 and most of the largest SINR metrics occur in the noise subspace, the performance of the SINR metric method is nearly identical in both cases. The eigenvalues of $\mathbf{R}$, the CSM, and $\sigma_d^2$ are plotted in Fig. 10. A comparison of Figs. 8 and 10 reveals that $\sigma_d^2$ and the CSM are approximately 35 dB less in Case 4 than in Case 1. The improved performance of the GSC methods can be attributed to this 35 dB difference. Since the SINR at a transformation rank of 1 is starting approximately 35 dB higher in Case 4, the addition of each CSM to the summation in (9) causes a greater change in the SINR performance in Case 4 than in Case 1.

The nearly identical SINR performance of the SINR metric method between Case 1 and Case 4 is also present between Cases 2 and 5 and Cases 3 and 6 (see Figs. 5 and 6). This similarity in SINR performance suggests that the SINR metric method is
nearly invariant to changes in the interference power level, which is not true for the CSM GSC method. We also expect the SINR performance of the SINR metric method to be relatively invariant to changes in the dimension of the interference subspace. That is, the basic shape of the SINR metric method performance curves will remain constant as the dimension of the interference subspace changes, but will shift up or down by an amount consistent with the change in the full dimension SINR performance. As the interference environment changes, the dimension of the interference and noise subspaces will change. However, the low rank nature of the interference plus noise correlation matrix should ensure that the eigenvalues of the noise subspace remain relatively constant, since these eigenvalues are essentially determined by the receiver noise power which should be constant for a given radar system. Recall that the numerator of the SINR metric is the projection of the steering vector onto a particular eigenvector. Thus, as long as the projections of the steering vector on the noise subspace do not change significantly as the interference environment changes, then the SINR metric method should be relatively invariant to the interference changes. The above argument assumes that the largest SINR metric occurs in the noise subspace which is not guaranteed.

Recall that as the steering (target) vector approaches the interference, the SINR performance of a STAP process will decrease. Now, notice that in Cases 1–3 and Cases 4–6 the steering (target) vector is moving towards the clutter ridge which passes through the zero Doppler and zero angle point of the normalized Doppler/angle plane since \( \beta = 1 \). Thus, Figs. 5 and 6 provide an indication of the SINR performance of the SINR metric and CSM GSC methods as the steering vector approaches the interference. An examination of Figs. 5 and 6 reveals...
the expected decrease in SINR performance, but also reveals that the basic shape of the SINR performance curves of the two methods do not change significantly as the steering vector approaches the interference.

In Case 1 (high clutter and jamming), we observe that the small method provides better performance than the CSM method below a rank of approximately 53 and approximately 3 to 4 dB less than the SINR metric method (see Fig. 3). As noted earlier, when the rank decreases below 77, the CSM method does not have sufficient degrees of freedom to cancel all the interference components. In contrast, the small method forces the weight vector to lie in the noise subspace effectively cancelling all the interference components. An examination of Fig. 7 reveals a wide separation in the SINR metric values associated with the noise and interference subspaces. Thus, the SINR is primarily determined by the noise subspace SINR metric values and leads the SINR metric method to compute a weight vector that also lies in the noise subspace. Therefore, the small and SINR metric methods provide similar performance, with the difference attributed to the ordering of the eigenvectors. Note that this similarity does not carry over to Case 4 (lower clutter and jamming, see Fig. 4), since the separation in the SINR metric values associated with the noise and interference subspace is decreased (see Fig. 9). Thus, the relative weight of each noise subspace SINR metric is decreased and the SINR builds up at a slower rate as a function of rank for the small method.

IV. CONCLUSION

We have extended the cost function concept used by Goldstein and Reed with the GSC to the direct form processor. The result is a new RR direct form processor, where the columns of the rank reduction transformation are selected as the eigenvectors of the interference plus noise correlation matrix based on the SINR metric. The SINR metric is used to identify the eigenvectors which minimize the loss in output SINR. We have presented simulation results that demonstrate the potential of SINR metric method under ideal conditions and highlight that the SINR performance of the proposed and CSM GSC methods are dependent on the interference plus noise environment and transformation rank. For a given interference plus noise environment, the simulation results show that the CSM GSC method outperforms the SINR metric method above a certain rank, while the converse is true below this rank. These results suggest the potential need for more than one implementation structure (method) in a STAP system, where the method is selected based on the scenario and available resources (e.g. secondary data support and computational power). Tools, such as the CSM and SINR metric, should prove to be invaluable in identifying and assessing candidate methods.

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