In this work, continuous-time systems are considered, and an innovative system identification approach is applied in the frequency domain, where phasor measurements, as provided by frequency analyzers, are used. This algorithm is obtained through a careful analysis of the physics of the measurement situation, following which the method of minimum variance (MV) estimation is iteratively applied in a linear regression framework. This process renders an efficient pseudominimum variance algorithm that is of linear prediction type and is superior to alternative methods. Thus, it is shown that the parameter estimate is a fixed point of a nonlinear mapping associated with the MV calculation. This is in contrast with the more conventional approaches, where the parameter estimate is sought as a solution of an optimization problem. Hence, an efficient pseudo MV algorithm is obtained that is of linear prediction type and is superior to alternative system identification. This parameter estimation algorithm is developed and validated in simulations which include measurement noise. Careful modeling of the stochastic estimation problem renders an efficient system identification algorithm.

Moreover, the proposed system identification algorithm yields an identified model and an estimate of the model uncertainty. The error is expressed in terms of the uncertainty of the coefficients of the transfer function of the plant, the latter being easily transformable into an expression of uncertainty about the physical parameters of the plant. In other words, the proposed system identification algorithm directly yields an identified model and its structured uncertainty, as opposed to plant uncertainty gauged with the $H_\infty$ norm.

The phasor approach is developed in Section II. A stochastic analysis of the system identification problem with measurement noise is performed in Section III. Section V explains how to calculate a needed covariance matrix for different assumptions about the measurement model. Section VI contains the results of applying the novel generalized MV (GMV) algorithm to a second-order system, as well as the comparisons with the other algorithms. The system used is representative of the short period dynamics of an aircraft. The performance of the different system identification algorithms is briefly discussed in Section VII, and concluding remarks are made in Section VIII.

II. FREQUENCY-DOMAIN-BASED PHASOR APPROACH

Consider the stable frequency domain transfer function, where the order of the numerator and
denominator are known:

\[ T(s) = \frac{y(s)}{u(s)} = \frac{b_1s^{n-1} + b_2s^{n-2} + \ldots + b_{n-1}s + b_n}{s^n - a_1s^{n-1} - a_2s^{n-2} - \ldots - a_{n-1}s - a_n}. \]

(1)

Here, the parameters \( a_1 \ldots a_n \) and \( b_1 \ldots b_n \) are unknown, the \( a_1 \ldots a_n \) coefficients are negative for a stable system, and the \( b_1 \ldots b_{n-1} \) coefficients equal zero when the order of the numerator is \( q \) less than the order of the denominator. If these parameters are equal to zero, this should be taken into account in the development. If one attempts to blindly identify all the parameters, assuming that the zero value parameters will just identify to zero, the results will suffer, for overmodeling is not a proposition in system identification.

By applying an input to this system of the form

\[ u_k(t) = \cos(\omega_k t) \]

and waiting for steady-state to be achieved, the steady-state output is

\[ y_k(t) = (A_k + jB_k)\cos(\omega_k t). \]

The following is a fundamental result of linear system theory

\[ A_k + jB_k = \frac{b_1(j\omega_k)^{n-1} + \ldots + b_n(j\omega_k)^n + b_{n-1}(j\omega_k) + b_n}{(j\omega_k)^n - a_1(j\omega_k)^{n-1} - \ldots - a_{n-1}(j\omega_k) - a_n} \]

which yields

\[ (A_k + jB_k)(j\omega_k)^n = \sum_{i=1}^{n} a_i(A_k + jB_k)(j\omega_k)^{n-i} + \sum_{i=1}^{n} b_i(j\omega_k)^{n-i} \]

Hence, a linear equation in the \( 2n \) \( a_i \) and \( b_i \) coefficients is obtained

\[ (A_k + jB_k)(j\omega_k)^n = \sum_{i=1}^{n} a_i(A_k + jB_k)(j\omega_k)^{n-i} + \sum_{i=1}^{n} b_i(j\omega_k)^{n-i} \]

which yields

\[ (A_k + jB_k) = \sum_{i=1}^{n} (A_k + jB_k)(j\omega_k)^{n-i} + \sum_{i=1}^{n} (j\omega_k)^{n-i}b_i. \]

Through algebraic manipulations, one obtains the linear system:

\[
\begin{bmatrix}
A_k
& B_k
& -A_k
& -B_k
& A_k
& \ldots
& 1
& 0
& \frac{1}{\omega_k}
& \frac{1}{\omega_k}
& \frac{1}{\omega_k}
& \ldots
& 0
& 0
& 0
& \ldots

\end{bmatrix}
\begin{bmatrix}
a_1
& b_1
& \ldots
& a_n
& b_n
\end{bmatrix}
= \begin{bmatrix}
-B_k
A_k
\end{bmatrix}.
\]

(2)

As can be seen, each sinusoidal test input produces two equations in the \( 2n \) unknowns. Therefore, \( n \) distinct sinusoids are needed to produce the \( 2n \) equations required for the determination of the parameter vector \( \theta = [a_1 \ldots a_n b_1 \ldots b_n]^T \in \mathbb{R}^{2n} \).

III. LINEAR-REGRESSION-BASED IDENTIFICATION

In reality, no physical measurements can be made without measurement noise, so we consider the linear regression equation from statistics:

\[ z = H\theta + \epsilon, \quad \epsilon = \mathcal{N}(0, \Sigma), \quad \Sigma > 0 \]

where the Gaussian random vector \( \epsilon \), referred to as “equation error,” represents measurement noise. In least squares (LS) based identification, the estimate \( \hat{\theta} \) is the vector that minimizes the square of the equation error vector. Based on the information above, the best linear unbiased estimate (BLUE) [3], or the estimate which has the MV out of all linear unbiased estimates (referred to as the MV estimate) is given by

\[ \hat{\theta}_{MV} = (H^T\Sigma^{-1}H)^{-1}H^T\Sigma^{-1}z. \]

(3)

For the special case when \( \Sigma = \sigma^2I \), the estimate \( \hat{\theta}_{LS} = \hat{\theta}_{LS}, \) where

\[ \hat{\theta}_{LS} = (H^T\Sigma^{-1}H)^{-1}H^Tz. \]

(4)

The primary drawback of LS estimation is the requirement that the equation error covariance \( \Sigma = \sigma^2I \), viz., lack of correlation. If it is not, as in the case when measurement noise is present and dynamical systems are considered, i.e., the traditional control situation, then there is correlation \( \Sigma = \sigma^2I \), and \( \hat{\theta}_{LS} \) will be biased. The GMV method properly accounts for the equation error covariance and thus reduces the bias, but \( \Sigma \) is often not known in practice. One of the main thrusts in this research is the correct modeling of \( \Sigma \), yielding parameter estimates with smaller bias.

For the phasor-based linear regression developed in Section II, one cannot obtain the true \( \theta \) and \( \Sigma \) phasor components. Rather, one has access to the noise-corrupted \( \theta_{kn} \) and \( \Sigma_{kn} \) measurements, where the
measured quantities are

\[ A_{kn} = A_k + v_{Ak}, \quad v_A = \mathcal{N}(0, \sigma_A^2) \]

\[ B_{kn} = B_k + v_{Bk}, \quad v_B = \mathcal{N}(0, \sigma_B^2). \]

Here, the random variables \( v_{Ak} \) and \( v_{Bk} \) are the measurement noises in the \( k \)th experiment, \( k = 1, 2, \ldots, m \). Therefore in reality, (3) is

\[ z_n = \frac{A_{kn}}{\omega_k} \quad \text{and} \quad \frac{B_{kn}}{\omega_k} - \frac{A_{kn}}{\omega_k} \begin{bmatrix} \omega_k^2 & \omega_k^3 & \omega_k^4 & \omega_k^5 \end{bmatrix} \begin{bmatrix} a_n & \cdots & a_n & b_1 & b_2 & \cdots & b_n \end{bmatrix}^T \]

This equation can now be written as a linear regression with

\[
\begin{bmatrix}
\begin{array}{cccccc}
\frac{A_{kn} - v_{Ak}}{\omega_k} & \frac{B_{kn} - v_{Bk}}{\omega_k} & \vdots & \vdots & \frac{-A_{kn} - v_{Ak}}{\omega_k} & 1 \\
\frac{-B_{kn} - v_{Bk}}{\omega_k} & \frac{A_{kn} - v_{Ak}}{\omega_k} & \vdots & \vdots & \frac{-B_{kn} - v_{Bk}}{\omega_k} & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
\theta \\
\epsilon
\end{bmatrix}
= \begin{bmatrix}
\begin{array}{c}
\frac{-B_{kn} - v_{Bk}}{\omega_k} \\

\end{array}
\begin{array}{c}
\frac{A_{kn} - v_{Ak}}{\omega_k} \\
\end{array}
\end{bmatrix}.
\]

This is now in a linear regression form as in (4), where the Gaussian random vector \( \epsilon \) is the equation

\[
\begin{bmatrix}
\epsilon_1 \\

\epsilon_2 \\
\vdots \\
\epsilon_m
\end{bmatrix}
= \begin{bmatrix}
T_1 v_1 \\
T_2 v_2 \\
\vdots \\
T_m v_m
\end{bmatrix}.
\]

Additionally, more than the minimal \( n \) sinusoidal test signals will be needed to overcome the measurement noise effects and obtain an accurate estimate. Hence, \( m (\geq n) \) measurements are taken, and the measurement error resulting from the measurement noise. The critical measurement noise covariance matrix \( \mathbf{R} \) is analytically determined by calculating the expectation \( E\{\epsilon \epsilon^T\} \). If the measurement noise is uncorrelated from
experiment to experiment \((E\{v_i v_j^T\} = 0, k \neq j)\), then \(R\) is reduced to a block diagonal matrix of the form

\[
R = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_m \end{bmatrix}_{2m \times 2m}
\]

where the \(2 \times 2\) \(R_k\) submatrix \((k = 1, 2, \ldots, m)\) is of the form

\[
R_k = E\{\epsilon_k \epsilon_k^T\} = E\{T_k v_i v_i^T T_k^T\}
\]

where

\[
R_{AB} = \begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix}.
\]

Note that

\[
\begin{bmatrix} R_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_m^{-1} \end{bmatrix}
\]

\[
H^T R^{-1} H = \sum_{k=1}^{m} H_k^T R_k^{-1} H_k,
\]

\[
R^{-1} z = \sum_{k=1}^{m} R_k^{-1} z_k
\]

so then explicit formulae are obtained

\[
\begin{align*}
\hat{\theta}_{GMV} &= \left( \sum_{k=1}^{m} H_k^T R_k^{-1}(\hat{\theta}_{GMV}) H_k \right)^{-1} \sum_{k=1}^{m} H_k^T R_k^{-1}(\hat{\theta}_{GMV}) z_k \\
P_{GMV} &= \left( \sum_{k=1}^{m} H_k^T R_k^{-1}(\hat{\theta}_{GMV}) H_k \right)^{-1}
\end{align*}
\]

IV. DEALING WITH CORRELATION

There are several methods in the literature today that were developed to overcome the correlation introduced by the dynamical model, thereby reducing the bias in the LS estimate. The two principal methods discussed here are the generalized least squares (GLS) and the instrumental variable (IV) methods [4]. However, these methods do not specifically address the particular correlation introduced by measurement noise in dynamical systems.

The method of GMV developed in this work is based on carefully modeling the measurement situation, and it addresses the particular correlation in the equation error introduced by measurement noise in dynamical systems.

A. Generalized Minimum Variance

To address the issue of measurement noise properly, the following procedure is proposed. The available information can be arranged in a linear regression equation given by

\[
z = H \theta + \epsilon \quad \epsilon = T(\theta) v, \quad \epsilon = N(0, R), \quad R_i \geq 0.
\]

To use insight gained from the MV identification method, it is necessary to obtain the equation error covariance matrix \(R\), where

\[
R = E[\epsilon \epsilon^T] = TR_i T^T.
\]

Unfortunately, \(R\) is not known \(a\ priori\), because in addition to the dependence on the measurement error \(\sigma_v\) of the given sensor, it is a function of the (as yet unknown) coefficients of the transfer function of the system, i.e., \(R = R(\theta)\). Two related, but different, estimation concepts can be derived from this MV-based approach.

First, one could minimize the associated cost function to obtain the estimate:

\[
\hat{\theta} = \arg \min_{\theta} \epsilon^T(\theta) \epsilon(\theta) = (z - H \theta)^T R^{-1}(\theta)(z - H \theta).
\]

Because of the dependence of \(R\) on \(\theta\), this leads to a complicated numerical search for a global minimum. In global nonlinear searches such as this, the appearance of local minima is virtually guaranteed as the noise levels increase. There is a problem with local minima in system identification problems that tends not to be a problem in other minimization problems. While a local minimum in a general minimization problem probably produces a solution with a lower cost, which may be acceptable as a solution in a design ephanization scenario since the cost has been lowered, the local minimum in a system identification problem can be very far from the required global minimum and is probably incorrect. Additionally, in general optimization problems, a local minimum with a cost that is very close to the global minimum cost probably produces an acceptable solution. However, the fact that the cost at a local minimum in a system identification problem is close to the global minimum cost has no bearing on the “closeness” of the estimate to the correct solution and is not a proper gauge of the estimation performance. It most likely results in a completely different, and incorrect, estimate. Therefore, only the global minimum is an acceptable solution in system identification.

In an attempt to avoid this complication, (5) is used to obtain the second possible derivation of the estimate. The GMV estimate is given by the point
\[ \hat{\theta}_{\text{GMV}} = (H^T R^{-1}(\hat{\theta}_{\text{GMV}})H)^{-1} H^T R^{-1}(\hat{\theta}_{\text{GMV}})z. \] (14)

Note that when \( R \) is known and is not \( \theta \) dependent, (14) yields the MV parameter estimate. When \( R \) is \( \theta \) dependent, as is the case in system identification, a fixed point problem formulation ensues. There are many different ways of searching for fixed points like \( \hat{\theta}_{\text{GMV}} \), but the following simple iterative algorithm has been effective in finding the correct fixed point for signal-to-noise ratios (SNRs) approaching 0 dB. This algorithm is motivated by the iteration for fixed points of contraction mappings, for which the existence of a fixed point is guaranteed [5].

**Step 1** Set \( i = 0 \) and calculate an initial parameter estimate using LS
\[ \hat{\theta}_i = \hat{\theta}_0 = (H^T H)^{-1} H^T z. \]

**Step 2** Calculate \( R(\hat{\theta}_i) \) using (12).

**Step 3** Calculate \( \hat{\theta}_{i+1} \) via (5)
\[ \hat{\theta}_{i+1} = (H^T R^{-1}(\hat{\theta}_i)H)^{-1} H^T R^{-1}(\hat{\theta}_i)z. \]

**Step 4** If \( \|\hat{\theta}_{i+1} - \hat{\theta}_i\| \) is less than some acceptable value, proceed to step 5. Otherwise, increment \( i \) and return to step 2.

**Step 5** Set \( \hat{\theta}_{\text{GMV}} = \hat{\theta}_{i+1} \).

**Step 6** The error covariance of the estimate \( \hat{\theta}_{\text{GMV}} \) is then given by
\[ (H^T R^{-1}(\hat{\theta}_{\text{GMV}})H)^{-1}. \] (15)

A fixed point for (14) does exist, at least for the problems of linear dynamical systems, as is shown in Appendix A. Also, the algorithm has converged within the numerical limits of Matlab for all problems examined thus far. The number of iterations required is quite small for small noise levels, but does increase as the noise level increases (the SNR decreases).

Problems do arise as the noise levels increase, as is common in system identification problems at low SNRs. In this work, it is common for one other fixed point to appear at higher noise levels, and possible methods of dealing with this are discussed in Section VIC. However, the other methods of identification used for comparison in this work tend to suffer at these high noise levels as well.

**B. Other Methods of Correlation Compensation**

There are many methods of compensating for correlated measurement noise, as discussed in [6], but the two used for comparison in this work are the IV method and the GLS method. These two methods, or variations thereof, are common in the literature.

For the IV method, one calculates the next estimate using an \( H \) regressor matrix which is the \( H \) matrix that would have resulted from a system consisting of the estimated parameters. In other words, the estimated parameters yield an estimated system. The elements of the \( \hat{H} \) matrix are then the transformed magnitude and phase values from the estimated Bode plot. The following equation is then iterated until a suitable convergence criterion is met:
\[ \hat{\theta}_{\text{IV}} = \left(H^T H\right)^{-1} \hat{H}^T z. \] (16)

Similarly, the GLS method exploits the form of \( \epsilon \) in (4). The contents of the \( z \) and \( H \) matrices are filtered through some form of \( T^{-1} \), and the LS estimate is calculated using these filtered results. The filtering tends to “whiten” the equation error, reducing the correlation and causing the LS formulation to be more correct. This is also an iterative procedure, as \( T \) is a function of \( \theta \), so the contents of \( z \) and \( H \) must be refiltered at each step.

A purported example of the GLS method is contained in [7]. The derivation of the noise model is essentially the same, but too many simplifying assumptions are made about the nature of the measurement noise. The final result can be shown to be identical to a GMV algorithm that assumes \( R_{AB} = I \).

In summary, the correlation in the equation error introduced by measurement noise in dynamical systems effectively eliminates the LS estimate as a valid result. Some method of dealing with this correlation must be found in order to obtain an accurate estimate. There are many current methods of dealing with the correlation, but none seem to specifically address the particular correlation introduced by measurement noise. The stochastic modeling and novel GMV identification concept described here are not limited to time-domain dynamical systems. They can also be applied to any system identification problem where certain measurements are correlated with one another.

**V. DETERMINATION OF \( R_{AB} \)**

The correct calculation of the \( R \) matrix is primarily dependent on the \( T \) matrix, but the \( R_{AB} \) matrix is also required. These matrices are critical for achieving good identification performance. If the measurement noise is on \( A_k \) and \( B_k \) directly, then this is not a problem. However, to make it more complicated, direct measurements of \( A_k \) and \( B_k \) are generally not available. Rather, Fourier analyzer supplied measurements of magnitude and phase angle are available. Therefore in this section, two different scenarios are discussed: 1) constant strength, uncorrelated noise on the real and imaginary phasor components directly, and 2) constant strength,
uncorrelated noise on the magnitude in decibels and phase angle in degrees as supplied by a frequency analyzer.

A. Measurement Noise on the Phasor Components $A_k$ and $B_k$

This is the most unlikely scenario of the three, but it is also the easiest and the one on which the derivation in Section III is based. Here, one assumes direct access to the real and imaginary parts of the frequency response data. This implies that the measurement noise strengths and correlation are known, i.e., $R_{AB}$ is known.

B. Noise on the Analyzer Supplied $M_{k\text{dB}}$ and $\phi_{k\text{deg}}$

As stated previously, direct measurements of $A_k$ and $B_k$ are generally not available. The more likely scenario is that one has hardware or software supplied measurements of magnitude ($M_k$) and phase angle ($\phi_k$). The following equations are then used to calculate $A_k$ and $B_k$

$$A_k = M_k \cos \phi_k, \quad B_k = M_k \sin \phi_k.$$  

(17)

It is more likely that the measurement noises ($\nu_{M_{\phi}}$) on the observables $M_k (\mathcal{N}(0,\sigma^2_{M}))$ and $\phi_k (\mathcal{N}(0,\sigma^2_{\phi}))$ are uncorrelated, and this would lead to correlation between $\nu_{A_k}$ and $\nu_{B_k}$. This also leads to another complication in that the strength of the noise on $A_k$ and $B_k$ is different at each frequency. To remedy this, one can assume that the noises are small, and using (17), the following linearized relation is derived

$$\begin{bmatrix} \Delta A_k \\ \Delta B_k \end{bmatrix} = \begin{bmatrix} \cos \phi_k & -M_k \sin \phi_k \\ \sin \phi_k & M_k \cos \phi_k \end{bmatrix} \begin{bmatrix} \Delta M_k \\ \Delta \phi_k \end{bmatrix}.$$  

Now $R_{AB}$ is calculated as follows

$$R_{AB} = E(\nu_{M_{\phi}}^T \nu_{M_{\phi}}) = T_{M\phi} E(\nu_{M_{\phi}}^T \nu_{M_{\phi}}) T_{M\phi}^T = T_{M\phi} R_{M\phi} T_{M\phi}^T.$$  

(18)

This indicates that $R_{AB}$ will be different for each measurement, and will need to be recalculated each time before adding $R_k$ to the block diagonal $R$ matrix. In practice, this calculation is not completely accurate, since the true values of $M_k$ and $\phi_k$, which are required for $T_{M\phi}$, are not available. One can use the measured values in their place and achieve good results for smaller noise values, but much better results are obtained if one uses the parameter estimates to estimate the transformation matrix as well.

The motivation for this next derivation arises from hardware and software considerations. Manufacturer’s accuracy specifications for the Tektronix frequency analyzer [1] are given in dB and degree values, but the cos( ) and sin( ) functions in Matlab require radian values. This is handled through the derivation of an additional linearized transformation. The pertinent equations are

$$M_k = 10^{M_{k\text{dB}}/20}, \quad \phi_k = \phi_{k\text{deg}} \frac{\pi}{180}.$$  

This leads to changes in $R_{M\phi}$ at each frequency, so once again assuming small values of noise, the following is derived

$$\begin{bmatrix} \Delta M_k \\ \Delta \phi_k \end{bmatrix} = \begin{bmatrix} 10^{M_{k\text{dB}}/20} & 0 \\ 0 & \frac{\pi}{180} \end{bmatrix} \begin{bmatrix} \Delta M_k \\ \Delta \phi_k \end{bmatrix}.$$  

This implies that

$$R_{M\phi} = E(\nu_{M\phi}^T \nu_{M\phi}) = T_{M\phi} R_{M\phi} T_{M\phi}^T$$  

(19)

so

$$R_{AB} = T_{M\phi} R_{M\phi} T_{M\phi}^T T_{M\phi}^T$$  

(20)

As before, the parameter estimates are used to estimate this transformation matrix as well. Additionally, in practice the $T_{M\phi}$ and $T_{M\phi}^T$ (and even $T_i$) transformations can be combined into one transformation matrix for the calculation of $R_k$.

VI. EXAMPLE

For investigative purposes, a general second-order dynamical system, which is representative of an aircraft’s elevator-to-pitch rate transfer function, is considered. The actual parameter values are obtained from [8], and the transfer function is given by

$$T(s) = \frac{y(s)}{u(s)} = \frac{b_1 s + b_2}{s^2 - a_1 s - a_2} = \frac{4.8 s + 1.44}{s^2 + 0.84s + 1.44}$$  

(21)

The Bode plot for this transfer function, which is representative of the pitch dynamics of an aircraft, and which is used for inner-loop flight control system design, is shown in Fig. 1. Also shown are the forty measurement frequencies used in each of the following experiments.

The first thing that needs to be done is to obtain the proper noise model. For this second-order system, the equation error noise vector in (8) is given by

$$e_k = \begin{bmatrix} -\frac{a_1 v_{b_k}}{\omega_k} - \frac{a_2 v_{b_k}}{\omega_k^2} - v_{b_k} \\ -\frac{a_1 v_{b_k}}{\omega_k} + \frac{a_2 v_{b_k}}{\omega_k^2} + v_{b_k} - \frac{1}{\omega_k^2} \left( -1 - \frac{a_1}{\omega_k^2} \right) \left( v_{b_k} \right) \end{bmatrix}.$$  

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whose elements are explicitly given by

\[
R_k = \begin{bmatrix}
-\frac{a_1}{\omega_k} & 1 - \frac{a_2}{\omega_k} \\
1 + \frac{a_2}{\omega_k^2} & -\frac{a_1}{\omega_k} \\
\end{bmatrix} \begin{bmatrix}
\sigma_A^2 & \sigma_{AB} \\
\sigma_{AB} & \sigma_B^2 \\
\end{bmatrix} \begin{bmatrix}
-\frac{a_1}{\omega_k} & 1 - \frac{a_2}{\omega_k} \\
1 + \frac{a_2}{\omega_k^2} & -\frac{a_1}{\omega_k} \\
\end{bmatrix}^T
\]

This yields the \(2 \times 2\) covariance matrix \(R_k\)

\[
R_{k11} = \sigma_A^2 \left( \frac{a_1^2}{\omega_k^2} \right) + \sigma_B^2 \left( \frac{2a_1a_2}{\omega_k^2} + \frac{2a_1}{\omega_k} \right) \\
+ \sigma_B^2 \left( \frac{2a_2}{\omega_k^2} + \frac{a_2}{\omega_k} + 1 \right)
\]

\[
R_{k22} = \sigma_B^2 \left( \frac{a_2^2}{\omega_k^2} \right) + \sigma_{AB}^2 \left( \frac{2a_1a_2}{\omega_k^2} - \frac{2a_1}{\omega_k} \right) \\
+ \sigma_A^2 \left( \frac{2a_2}{\omega_k^2} + \frac{a_2}{\omega_k} + 1 \right)
\]

\[
R_{k12} = \sigma_A^2 \left( \frac{a_1}{\omega_k} \right) + \sigma_{AB}^2 \left( \frac{a_1^2}{\omega_k^2} - \frac{a_2}{\omega_k^2} - \frac{2a_2}{\omega_k} - 1 \right) \\
+ \sigma_B^2 \left( \frac{a_1}{\omega_k} + \frac{a_2}{\omega_k} \right)
\]

\[R_{k} = \begin{bmatrix} R_{k11} & R_{k12} \\
R_{k21} & R_{k22} \end{bmatrix}
\]

Section VIC discusses the identification results as the measurement noise strength is increased by powers of 10, i.e.,

\[
R_{ABnew} = 10^p \cdot R_{AB} \quad p = 0, 1, 2, 3, 4.
\]

B. Inadequacy of the Least Squares Method

In this section, one pretends that \(A_k\) and \(B_k\) are directly measurable. To compare the different estimation methods, uncorrelated and equal strength noises are added to the true \(A_k\) and \(B_k\) after they are computed from a Bode analysis of the transfer function in (21). Representative noise with values of \(\sigma_A = \sigma_B = 0.023\). This choice of \(R_{AB}\) results in a diagonal \(R\) matrix, but it is still not a scalar multiple of the identity matrix because of the varying measurement frequencies.

Fig. 2 displays the results of a 100 run Monte-Carlo (MC) analysis, which upon evoking the weak law of large numbers [9], renders a gauge of the estimation bias of the identification algorithm. The estimation results are first normalized by dividing each estimate by the true estimate, and then plotted. Ellipses are plotted representing the actual one sigma variation of each estimation method. The axes of the ellipses intersect at the average estimate for each method. The algorithm predicted estimation error covariances are not plotted, because they are not known for the LS or GLS methods, i.e., (6) cannot be calculated because there is no constant \(\sigma\) that can be used. Indeed, the \(\sigma\) required in (6) is estimated as follows:

\[
\hat{\sigma} = \sqrt{\hat{z}^T \hat{z} / (m - 4)}, \quad \hat{z} = z - \hat{H}\hat{\theta}.
\]

However, this estimate is data driven and the predicted estimation error covariance still would not be an
Fig. 2. All estimates with uncorrelated $A_k$ and $B_k$ noise.

TABLE I
Numerical Results for Noise on $A$ and $B$

<table>
<thead>
<tr>
<th>Method</th>
<th>$\theta$</th>
<th>$\epsilon$ (%)</th>
<th>$\sigma_e$</th>
<th>$\sigma_p$</th>
<th>$\epsilon$ (%)</th>
<th>$\sigma_e$</th>
<th>$\sigma_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least Squares</td>
<td>$a_1$</td>
<td>-15.0562</td>
<td>7.7862 n/a</td>
<td>-65.0425</td>
<td>14.9841 n/a</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b_1$</td>
<td>-14.1309</td>
<td>5.5541 n/a</td>
<td>-58.5484</td>
<td>11.4132 n/a</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b_2$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Minimum</td>
<td>$a_1$</td>
<td>-0.0206</td>
<td>0.2690 0.2965</td>
<td>-0.1449</td>
<td>0.8501 n/a</td>
<td>-0.0265</td>
<td>0.1816 0.1764</td>
</tr>
<tr>
<td>Squares</td>
<td>$b_1$</td>
<td>-0.0017</td>
<td>0.2390 0.2355</td>
<td>-0.0520</td>
<td>0.7538 n/a</td>
<td>-0.0554</td>
<td>0.5991 0.6094</td>
</tr>
<tr>
<td>Generalized Least</td>
<td>$a_1$</td>
<td>-0.0206</td>
<td>0.2690 n/a</td>
<td>-0.1449</td>
<td>0.8501 n/a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Squares</td>
<td>$b_1$</td>
<td>-0.0017</td>
<td>0.2390 n/a</td>
<td>-0.0520</td>
<td>0.7538 n/a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Instrumental Variable</td>
<td>$a_1$</td>
<td>0.7340</td>
<td>5.3380 11.0606</td>
<td>3.4853</td>
<td>17.2366 n/a</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>$a_2$</td>
<td>-0.1310</td>
<td>1.4472 6.1274</td>
<td>-0.1748</td>
<td>4.7678 n/a</td>
<td>-0.2866</td>
<td>2.3113 2.6029</td>
</tr>
</tbody>
</table>

Dramatic improvements can be made by using one of the other estimation methods. As seen in the plot, the IV method does have quite a large sigma value, but the estimate is not nearly so biased as the LS. The GLS and GMV estimates cannot be distinguished on this plot because they are tightly clustered around the true estimate. Fig. 3 zooms in to give a better look at these estimates. This magnification provides a good view of the IV estimates and sigma, but the GLS

accurate representation of the true error covariance. The available algorithm predicted covariances, along with all other numerical results, are given in Table I.

As can be seen in the plots, there are large biases in the LS average estimate, and the true parameter is well outside the one sigma bounds. This indicates, as is shown in the plot, that the majority of the parameter estimates in the hundred runs are further from the true estimate than one sigma.
and GMV estimates are still clustered rather tightly around the true parameter. Indeed, this is always the case. The GLS and GMV estimates are always better than the LS and IV. For this reason, the LS and IV estimates are dropped from most plots in the sequel. The numerical results are still included in the Appendix.

Further magnification in this noise scenario would provide little information on the comparison of the GLS and GMV estimates because they are identical in this case. The only difference between the two is a scaling of the $R$ matrix which cancels out in the final equation. The only thing gained by the GMV method is an accurate estimation of the estimation error covariance. As is seen in Table I, these covariance estimates are rather close to the actual ones. The small biases and covariance differences imply that the GMV and GLS system identification algorithms yield relatively unbiased parameter estimates. Additionally, GMV is doing a good job of predicting the accuracy of its estimate, i.e., the algorithm is “efficient.”

C. Reducing the SNR

The next portion of the experiment involves increasing the level of the measurement noise, as in (24). The results for $p = 1$ are very similar to $p = 0$. But when $p$ is increased to 2, a trend that causes problems later on starts to appear. As the strength of the measurement noise increases, the LS estimates start to cluster around the origin (Fig. 4). This may seem like it should not effect the GMV estimate, and it does not in this case, but recall that the GMV estimate is initialized with this LS estimate. When $p$ is increased to 3, the GMV estimates also start to cluster around 0. This is not a gradual migration like the one that happens in the LS estimates as the noise increases; it is sudden. Indeed, it is caused by the appearance of a second fixed point close to the origin.

To better illustrate this problem, a “shotgun approach” is taken for one particular noise realization. The results of the GMV algorithm, when it is initialized with a series of different vectors whose elements range from $-0.5$ to $0.5$, are shown in Fig. 5. The left figure shows the paths taken during the iterations. The first iteration step is shown by a dotted line, while the remaining iterations are plotted as a solid line. The end points (or fixed points) are shown as an $\ast$. Most of the initial value choices converge to the desired estimate, but several runs initialized close to the origin converge to a different value. This seems to be a detrimental property of the phasor-oriented GMV algorithm at high noise. How does one choose the “correct” fixed point? The solution to this may lie in the right-hand plot. This is a plot of $[f(\theta) - \theta]^T [f(\theta) - \theta]$ at each iteration. The
Fig. 5. Convergence properties for uncorrelated $A_k$ and $B_k$ noise.

Fig. 6. Convergence rates for LS initialization. Slowly converging paths in this plot correspond to those paths that are converging to the wrong point. Although the LS estimate is a convenient point to start the GMV algorithm, it is not the wisest choice, since it tends to approach the origin as the noise level increases.

A better choice would be a point in the same quadrant as the true parameter, but with a larger magnitude. This is usually not possible, because one does not know the true parameter. However, if the problem is known to be affected by high noise, one can choose several starting points, perhaps one from each quadrant, and the one that converges the fastest is the correct one.

If one examines the convergence rates for the $p = 3$ estimates in Fig. 4, which are shown in Fig. 6, it can be seen that many estimates converge very slowly, while others do not appear to converge at all. However, when the GMV algorithm is initialized at the point $\theta_0 = [-100 \ -100 \ 0 \ 0]^T$, the results improve dramatically. These results are shown in Fig. 7. As can be seen, the GMV estimates are no longer trapped around the origin, and the convergence rates are much quicker as well.

D. Noise on Observables

In this section, the noise is added to the $dB$ magnitude and phase angle (in deg), as is given in the Tektronix [1] specifications. Then, the two required transformations ((18) and (20)) are performed to obtain the correct $\sigma_A^2$, $\sigma_B^2$, and $\sigma_{AB}$ used for the $R_k$ matrix calculation in (22). Through proper modeling of the noise transformations, the GMV algorithm achieves better estimates than any of the other methods. The GLS algorithm comes the closest, and a comparison of these for the $p = 0$ case is shown in Fig. 8.

As can be seen, the GMV estimates have both a lower bias and error covariance than the GLS estimates. The reason for this is the proper modeling of the $R_k$ matrix. Initial attempts at this used the measured values in the transformation matrices. While this yields good results for small noise strengths,
the measured values for higher noise strengths do not provide an accurate estimate of the magnitude and phase required for the transformation matrix calculations. Problems arise in poor numerator estimates, and the parameters start to get trapped around the origin sooner than they would otherwise.

Instead, the measured values are used only for the first five GMV iterations. This is only done to get a more accurate estimate of the Bode plot, and may not even be necessary. After the fifth iteration, the GMV parameter estimates are used to estimate the Bode plots, and the magnitude and phase values from this estimation are used in the transformation matrices. This method produces much more accurate estimates, and it is also more robust to the "zero trapping" than before.

Higher noise strength runs were also performed for this scenario, and the numerical results are given in

<table>
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<tr>
<th>Method</th>
<th>( \theta )</th>
<th>( \epsilon ) (%)</th>
<th>( \sigma_\theta )</th>
<th>( \sigma_\epsilon )</th>
<th>( \sigma_\theta )</th>
<th>( \sigma_\epsilon )</th>
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<tbody>
<tr>
<td>Least Squares</td>
<td>( a_1 )</td>
<td>-2.9445</td>
<td>2.4267</td>
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<td></td>
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<td>1.2616</td>
<td>6.2037</td>
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<td>1.5155</td>
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<td>6.8051</td>
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<tr>
<td>Variance</td>
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<td>165.4659</td>
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</table>

Fig. 8. Estimates with uncorrelated \( M_{\text{dB}} \) and \( \phi_{\text{deg}} \) noise.

TABLE II
Numerical Results for Noise on \( R_{\theta} \) and \( \phi_{\text{deg}} \)
Table II. As before, the GMV results are the best, and the error covariance prediction is close to the actual.

VII. DISCUSSION

The normalized numerical results of all the experiments are summarized in Tables I and II in Appendix B. In all cases, the values shown correspond to a forty measurement linear regression. The MC averaged estimation error (ε) and sigma (σε) are given for all cases, and the algorithm predicted sigma (σp) is given for the MV and IV experiments. In each case, the bias in the MV estimate is about two orders of magnitude smaller than in the naive LS estimate, and the estimation error covariance is much smaller as well.

For p < 2, all 100 of the estimated Bode plots are virtually indistinguishable. Appendix B contains the final estimation results for p = 2, 3. The top plots show the true Bode plot and the average of the estimated Bode plots, along with the range of the estimated Bode plots. As can be seen, the spread is quite small even for p = 2. It appears, however, that the admittedly small, identification provided, uncertainty is well within the system’s bandwidth, i.e., it is structured uncertainty.

The bottom plots show the true poles and zero, along with the estimated poles and zeros. For p < 2, the estimated poles lie practically on a horizontal line, and even at p = 2, the vertical variation is very small, i.e., the identification algorithm renders a very accurate estimate of the damped natural frequency. A noticeable bias begins to appear when p reaches 3, but it is not all that large, considering the noise level.

VIII. CONCLUSIONS

In this paper, a frequency domain approach to the identification of continuous-time control systems using phasors as provided by frequency analyzers is taken. Gaussian measurement noise is assumed, as is customary in classical filtering and system identification work. The proper MV estimate equations are derived and are iteratively applied to the nonlinear estimation problem on hand. It is shown that the parameter estimate is a fixed point of a nonlinear mapping associated with the calculation of the MV parameter estimate, and is not the solution of an optimization problem, as is commonly proposed in the literature.

The results are first compared with the simple-minded LS estimate for a second-order system that is representative of an aircraft’s pitch dynamics, which is used for inner-loop flight control system design. The GMV estimate outperformed the LS estimate in all cases. The LS estimate does, however, provide a useful, albeit dangerous, starting point for iterating the GMV estimate. At small noise levels, it does not matter where the GMV algorithm is initialized, but at higher noise levels, it is dangerous to initialize the algorithm close to the origin. There is an additional fixed point appearance there that has proven capable of trapping the estimate. On a positive note, when the estimate is trapped, the algorithm takes much longer to converge, so using this failure signature, multiple high magnitude starting points should be used to allow one to find the desired fixed point. In conclusion, the LS estimate is not an effective one, even in cases of small measurement noise. Next, the performance of the novel fixed-point-based GMV system identification method is compared with alternative identification methods. Although the IV estimate is somewhat better than the LS estimate, it still performs poorly under increased noise levels. The GLS and GMV provide much more accurate estimates, and the GMV algorithm seems to provide lower biased and smaller covariance estimates in each case.

Thus, careful stochastic modeling of the estimation problem at hand renders an efficient identification algorithm for continuous-time systems using phasors. However, the experimental phasor approach is applicable to stable systems only.

APPENDIX A. EXISTENCE OF FIXED POINT

The primary problem in solving for the GMV estimate is that R is a nonlinear function of θ. Therefore, define f : R^{2n} → R^{2n},

\[ f(\theta) = [H^T R^{-1}(\theta) H]^{-1} H^T R^{-1}(\theta) z \]  

and use an iteration to search for a fixed point in R^{2n} where θ = f(θ). In searching for a fixed point, it would be nice to know that a fixed point actually exists. If it could be determined that f(θ) is a continuous, bounded function, then it would be easy to show that at least one fixed point exists.

A. Singularities and Continuity

The first step in showing that f(θ) is bounded is to show that it is a continuous function, or that it has only removable singularities. To do this, first return to the equation error noise vector in (9). Because sums, products, and inverses of rational functions are rational functions, and T_k is a matrix of rational functions in the parameters, then f(θ) is a rational function. Therefore, f(θ) is continuous everywhere it is defined. However, f(θ) is not defined everywhere, but has singularities where R is not invertible. Since R_{AB}, is, by definition, positive definite, the singularities occur when the determinant of any particular T_k matrix evaluates to zero. Since the transformation matrix T_k is affine in the denominator parameters, it


Insight can now be gained by looking at the physical nature of the problem. The singularities occur when \( T_k = 0 \), which implies that \( \epsilon = 0 \) in (9), i.e., there is no uncertainty in that measurement. For \( n = 2 \), this would mean \( z_k = H_{atk} \theta \), and a solution can be found by simple LS, i.e.,

\[
\lim_{\theta \to \tilde{\theta}} f(\theta) = \left[ H_k^T H_k \right]^{-1} H_k^T z_k. \tag{27}
\]

Unfortunately, this formulation is only valid for \( n \leq 2 \), because for \( n > 2 \), \( H_k^T R_k^{-1} H_k \) only has rank 2 and is therefore not invertible.

However, one can expand on the notion of a noiseless measurement, and use the Kalman filter update equations to add an additional measurement to a previous estimate. For \( n = 2 \) and \( n = 3 \), it is readily apparent that only one \( T_k \) can be equal to 0 for a given experiment. For \( n = 2 \), the singularities are points corresponding to \( a_2 = -\omega_k^2 \), and since two measurements cannot be taken at the same frequency, the points cannot be repeated. For \( n = 3 \), the individual singularity lines lie in the plane \( \omega_k^2 \), and again, since no two measurements can be taken at the same frequency, the singularity lines cannot intersect. It is suspected that this is the case for higher \( n \) as well.

This implies that there is only one “noiseless measurement” at any given point. In other words, only one term in each of the summations is undefined at any given singular point:

\[
H_k^T R_k^{-1} H_k \quad \text{and} \quad H_k^T R_k^{-1} z_k.
\]

These terms are not defined because \( R_k = 0 \), causing \( T_k = 0 \). Taking the Kalman filtering approach, let the preupdate covariance and estimate be the covariance and estimate calculated by using all the terms except the \( k \)th one

\[
\overline{P} = \left( \sum_{k \neq k} H_k^T R_k^{-1} H_k \right)^{-1} \quad \text{and} \quad \overline{\theta} = \overline{P} \sum_{k \neq k} H_k^T R_k^{-1} z_k.
\]

Then following the formulation in [4], it can be shown that in any neighborhood around the singularity,

\[
f(\theta) = \left( \sum_{k = 1}^{m} H_k^T R_k^{-1} H_k \right)^{-1} \left[ \sum_{k = 1}^{m} H_k^T R_k^{-1} z_k \right]
= [\overline{P} + H_k^T R_k^{-1} H_k]^{-1} [\overline{P} \overline{\theta} + H_k^T R_k^{-1} z_k]
= [\overline{P} - \overline{P} H_k^T (R_k + H_k \overline{P} H_k^T)^{-1} H_k \overline{P}] \times [\overline{P}^{-1} \overline{\theta} + H_k^T R_k^{-1} z_k]
\]

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\[
\sum_{k=1}^{m} H_k^T \left( \lim_{\alpha \to \infty} \frac{R_k}{\alpha^2} \right) H_k \right]^{-1} \times \left[ \sum_{k=1}^{m} H_k^T \left( \lim_{\alpha \to \infty} \frac{R_k}{\alpha^2} \right) z_k \right].
\]

This reduces the problem to finding the limit of \( R_k / \alpha^2 \) as \( \alpha \) approaches infinity, which is a significantly simpler problem, since
\[
\lim_{\alpha \to \infty} \frac{R_k}{\alpha^2} = \left( \lim_{\alpha \to \infty} \frac{T_k}{\alpha} \right) R_{AB} \left( \lim_{\alpha \to \infty} \frac{T_k}{\alpha} \right)^T.
\]

But here,
\[
T_k = A_0 + \alpha \sum_{j=1}^{n} A_{jk} h_j
\]
so
\[
\lim_{\alpha \to \infty} \frac{T_k}{\alpha} = \lim_{\alpha \to \infty} \left( \frac{A_0}{\alpha} + \sum_{j=1}^{n} A_{jk} h_j \right) = \sum_{j=1}^{n} A_{jk} h_j
\]
which leads to
\[
\lim_{\alpha \to \infty} \frac{R_k}{\alpha^2} = \left( \sum_{j=1}^{n} A_{jk} h_j \right) R_{AB} \left( \sum_{j=1}^{n} A_{jk} h_j \right)^T = R_{k\infty}
\]
and
\[
\lim_{\alpha \to \infty} f(\theta) = \left[ \sum_{k=1}^{m} H_k^T R_{k\infty}^{-1} H_k \right]^{-1} \sum_{k=1}^{m} H_k^T R_{k\infty}^{-1} z_k \equiv f_{\infty}(h).
\]
Since for any given \( h \), \( f_{\infty}(h) \) is simply the solution to a weighted LS problem, it is therefore a finite vector, and
\[
\| f_{\infty}(h) \|_\infty < \infty.
\]
Now let
\[
m(\alpha) = \max_{h \in H} f(\alpha h).
\]
From [10], it is known that \( m(\alpha) \) is a continuous function of \( \alpha \). Using the vector infinity norm,
\[
\| m(\alpha_1) - m(\alpha_2) \| = \| \max_{h} f(\alpha_1 h) - \max_{h} f(\alpha_2 h) \| \\
\leq \max_{h} \| f(\alpha_1 h) - f(\alpha_2 h) \| \\
< \epsilon \quad \text{for} \quad \| \alpha_1 - \alpha_2 \| < \delta.
\]
Also,
\[
\lim_{\alpha \to \infty} m(\alpha) = \max_{h \in H} f_{\infty}(h) < \infty.
\]
Since \( m(\alpha) \) is a continuous function bounded at infinity, so is its norm, and the supremum of that function is finite, that is,
\[
\sup_{\theta} \| f(\theta) \| = \sup_{\alpha} \| m(\alpha) \| = S < \infty.
\]
Since the supremum of the infinity norm of \( f \) is \( S \), this implies that \( f \) maps the convex, bounded
set $B(0, S^{2n}) \subset \mathbb{R}^{2n}$ into itself. Now, since $f(\theta)$ is a continuous function that maps a convex compact set into a convex compact set, then as a consequence of the Schauder Fixed Point Theorems [2], there exists a point $\overline{\theta} \in B(0, S^{2n})$ such that $f(\overline{\theta}) = \overline{\theta}$, i.e., a fixed point exists.

APPENDIX B. NOISE PLOTS AND RESULTS

This Appendix contains plots of representative noise, plots of estimated poles and Bode plots, and numerical results for the frequency domain experiments in this paper. Figs. 10 and 11 show the added and transformed noise on the magnitude in dBs and phase angle in degrees. The normalized numerical results for each of the examined noise scenarios are given in Tables I and II. Shown are the calculated average error in the estimate $e = \hat{\theta} - \theta$, the estimation error sigma $\sigma_e$, and the algorithm predicted error sigma $\sigma_p$ for the MC analysis. The n/a entries in the tables indicate that these methods do not produce an estimate for the covariance value. The other figures contain the final estimated Bode plots and poles and zeros for the higher noise cases (Figs. 12 and 13). Shown are the true and average estimated Bode plots, along with the one sigma bounds on the estimated Bode plots. The pole and zero figures show each of the estimated poles and zeros, along with the true values.
REFERENCES

Fig. 12. Final Bode plots and roots for \( M_{dB} \) and \( \phi_{deg} \) noise, \( p = 2 \).

*Introduction to Minimax.* 

Fig. 13. Final Bode plots and roots for \( M_{dB} \) and \( \phi_{deg} \) noise, \( p = 3 \).

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