Detection of Chi-Square Fluctuating Targets in Arbitrary Clutter

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In a recent paper, general expressions were derived for the density and cumulative probability functions of the amplitude of a linear matched-filter output given a nonfluctuating target in a clutter-limited environment. These expressions were based on the clutter amplitude density function. The results are extended to calculate the cumulative probability function of the output of a linear matched filter used to detect a chi-square fluctuating target in a clutter-limited environment. The resulting method is applied to a common radar clutter model, and experimental sonar data.

I. INTRODUCTION

The most important objective in the development of a practical echo-location system is the reliable detection of a target in clutter. The classical theory of detection—that based on the detection of a fluctuating or nonfluctuating point target embedded in Gaussian clutter and noise—is quite mature. However, in practice, it is often the case that neither target nor clutter displays statistical fluctuations that match those assumed in the derivation of the classical theory. If this is the case, then use of the detection thresholds provided by the classical theory can yield unacceptable performance of an echo-location system; i.e., the system will display a false-alarm rate higher than that predicted by the classical theory.

In radar, the matched-filter response to non-Gaussian clutter has been modeled by the Weibull, log-normal, and $K$-type probability density functions. In these cases, an approach to deriving the probability density function of the matched-filter output is to model the clutter as a compound-Gaussian random process [1]. In sonar, however, it has been found that the matched-filter response to clutter may not follow any of these statistical models; thus, other closed-form density functions must be used [2, 3], or must be empirically derived from data. If the later approach is used, any effect by system noise would be included in the empirically derived density function, although clutter often dominates the detection problem. In either case, a compound-Gaussian clutter model may not be easily found in closed form. It is then expedient to use a method for calculating detection probabilities that directly uses the probability density function for the matched-filter response to clutter.

We begin our derivation of the echo-location detection problem as follows. If $g(t)$ is the transmitted signal whose energy is $E$, then we model the return waveform as

$$x(t) = r_T e^{j\theta_T} g(t - \tau)e^{j\omega t} + c(t)$$

where $c(t)$ is the clutter return, and $r_T$ and $\theta_T$ are, respectively, the amplitude and phase of the reflection produced by a target whose range delay is $\tau$, and motion-induced Doppler shift in radian frequency is $\omega$. As shown in Fig. 1, the return is processed by a matched-filter receiver that calculates the integral correlation between the return and an energy-normalized ideal version of the transmit signal subjected to a hypothesized range delay \( \hat{\tau} \),

![Fig. 1. Matched-filter receiver structure.](image-url)
and Doppler shift $\hat{\omega}$. The output of the correlator is followed by an envelope detector to yield the receiver output $|\ell(\hat{\tau},\hat{\omega})|$. In practice, one calculates the value of $|\ell(\hat{\tau},\hat{\omega})|$ over a range of hypothesized range delays and Doppler shifts at discrete points or “bins” in range-Doppler space. However, our analysis assumes we are addressing the bin occupied by the target. In this case, the matched-filter output prior to envelope detection is the complex value

$$\ell = re^{i\theta} = r re^{i\theta} + r e^{i\theta_c}$$  \hspace{0.5cm} (2)

where $r_C$ and $\theta_C$ are, respectively, the amplitude and phase of the matched-filter response to the clutter. It is assumed $r_T$, $r_C$, $\theta_T$, and $\theta_C$ are independent random variables, and $\theta_T$ and $\theta_C$ are uniformly distributed in the interval $[0,2\pi]$. To calculate detection probabilities, one must determine the cumulative probability function of $r$.

For a fixed value of $r_T$, the cumulative probability function of $r$ in the presence of a nonfluctuating target in clutter is denoted by $F_{R,T+T}(r)$. Thus,

$$F_{R,T+T}(r) = \int_{r_T}^{\infty} f_{R,T+T}(r)f_R(r_T)dr_T$$  \hspace{0.5cm} (3)

where $f_R(r_T)$ is the density function describing the target amplitude fluctuation. This formulation works well if $F_{R,R,T+T}(r)$ is known in closed form, but in most practical cases it is not. One might know the target amplitude density function $f_{R,R,T+T}(r)$, but again its integral may not be known in closed form. Hence, one must evaluate the double integral

$$F_{R,T+T}(r) = \int_0^\infty \left[ \int_0^{r_T} f_{R,R,T+T}(a)da \right] f_{R,T}(r_T)dr_T.$$  \hspace{0.5cm} (4)

Moreover, if the clutter amplitude is not Rayleigh, then $f_{R,R,T+T}(a)$ is derived from an integral expression [4–6]. This implies that the calculation of $F_{R,T+T}(r)$ requires a triple integration.

The problem addressed here is the derivation of an integral expression for the cumulative probability function of the output of a matched filter that is used to detect a target buried in arbitrary clutter, and whose target strength $w$ is modeled as a chi-square random variable [7, 8] with a density function of

$$f_w(w) = \frac{N^N}{2^N\Gamma(N)\sigma_0^N}w^{N-1}e^{-w/2\sigma_0^2}$$  \hspace{0.5cm} (5)

where $N$ is any positive real number. In particular, $N = 1$ corresponds to a Rayleigh fluctuating target. $N = 2$ corresponds to a one-dominant-plus-Rayleigh fluctuating target, and $0 < N < 1$ corresponds to the Weinstock fluctuating target models [9]. The solution proceeds from (3), and yields an integral expression for $F_{R,T+T}(r)$ requiring only the clutter amplitude density function $f_{R,C}(r)$. Furthermore, the expression can eliminate the necessity of a double integration.

Although the problem has been stated in terms of the target strength, (3) and (4) dictate that we formulate the problem in terms of the target amplitude $r_T$. This is accomplished by invoking the transformation of random variable $r_T = \sqrt{w}$, implying the density function for $r_T$ is

$$f_{R,T}(r_T) = \frac{N^N}{2^N\Gamma(N)\sigma_0^N}r_T^{2N-1}e^{-r_T^2/2\sigma_0^2}$$  \hspace{0.5cm} (6)

where

$$E\{r_T^2\} = \frac{\Gamma(N+1/2)}{\Gamma(N)}\sqrt{\frac{2\sigma_0^2}{N}}$$  \hspace{0.5cm} (7)

$$E\{r_T^2\} = 2\sigma_0^2.$$  \hspace{0.5cm} (8)

Note that (8) is the expected target strength. This density function is known as the Nakagami fluctuation model [10].

The remainder of the paper is arranged as follows. In Section II, we derive an integral expression for the cumulative probability function requiring only the probability density function of the clutter amplitude at the matched-filter output. In Section III, several series expressions for the integration kernel are derived. In Section IV, we discuss two approaches to evaluating the integration kernel. Finally, in Section V, we present two numerical examples; one using a common radar clutter model, and the other using experimental sonar data.

II. THE CUMULATIVE PROBABILITY FUNCTION

In a recent paper, Drumheller [11] presented a method for calculating the density and cumulative probability functions for the matched-filter output amplitude $r$ in (2) assuming the target amplitude $r_T$ was nonrandom (nonfluctuating). The method was based on a technique developed by Jakeman and Pusey [12], and Jao [13, 14], and involved calculating the moment-generating functions for the joint cumulative probability functions of the real and imaginary parts of $r_T e^{i\theta_T}$ and $r_C e^{i\theta_C}$. The result was a set of integral expressions that used the clutter amplitude density function $f_{R,C}(r)$ in their integrand, and yielded either the matched-filter output amplitude density function $f_{R,T+T}(r)$, or cumulative probability function $F_{R,T+T}(r)$ through numerical integration. Hence, one did not need the clutter amplitude cumulative probability function, a function that might not be known in closed form, to calculate detection probabilities.
It can be shown that the cumulative probability function for the matched-filter output given the presence of a nonfluctuating target in arbitrary clutter is given by ([11, eq. (28)])

\[
F_{R,T+C}(r) = r \int_0^\infty F_{R,C}(a) \left[ \int_0^\infty J_0(\rho a)J_0(\rho r)J_1(\rho r) d\rho \right] da
\]

(9)

where \( J_0 \) and \( J_1 \) denote the Bessel functions of order 0 and 1. Taking the expectation of the cumulative probability function in (9) with respect to the density gives in (6) yields

\[
F_{R,T+C}(r) = \mathbb{E} \left\{ F_{R,T+C}(r) \right\}
\]

\[
= r \int_0^\infty F_{R,C}(a) \left[ \int_0^\infty J_0(\rho a)E\{J_0(\rho r)\} J_1(\rho r) d\rho \right] da
\]

(10)

where we have used the formula ([15, p. 716, eq. 6.631.1])

\[
\frac{N^N}{2^{N-1}\Gamma(N)\sigma_0^{2N}} \int_0^\infty J_0(\rho r) r^{2N-1} e^{-\rho^2/2\sigma_0^2} dr
\]

\[
= _1F_1(N;1;-\sigma_0^2 r^2/2N)
\]

(11)

with \(_1F_1\) denoting the confluent hypergeometric function, and have defined

\[
I(a \mid r, \sigma_0, N) = \int_0^{\infty} J_0(\rho a)J_1(\rho r) \cdot _1F_1(N;1;-\sigma_0^2 r^2/2N) d\rho.
\]

(12)

The main result is the last line of (10). It shows the matched-filter output amplitude cumulative probability function can be derived using only the clutter amplitude density function, and an integration kernel \( I(a \mid r, \sigma_0, N) \) as defined in (12). In the following section we show that the kernel has several alternate representations. Under some circumstances their use implies that only a single numerical integration of (10) is needed to calculate \( F_{R,T+C}(r) \).

III. ALTERNATE KERNEL REPRESENTATIONS

Calculation of \( F_{R,T+C}(r) \) could be accomplished by a numerical integration of (12); however, there are two problems with this approach. First, the integrand of (12) is oscillatory. This is undesirable, since numerical quadrature algorithms do not always yield accurate approximations to the integrals of oscillatory functions. Second, for some values of \( r, \sigma_0, \) and \( N \) the “envelope” of the integrand decays slowly. This makes it difficult to determine an appropriate finite upper limit of integration that will yield an acceptable approximation to \( I(a \mid r, \sigma_0, N) \).

To avoid these problems, we need to find either an alternate functional representation of \( I(a \mid r, \sigma_0, N) \), or an alternate integral representation whose integrand is better behaved. To that end, we begin by recognizing that the confluent hypergeometric function in the integrand of (12) has the Barnes integral representation ([16, p. 506, eq. 13.2.9])

\[
_1F_1(N;1;-\sigma_0^2 r^2/2N) = \frac{1}{2\pi i} \int_C \Gamma(N+s)\Gamma(-s) \left( \frac{\sigma_0^2 \rho^2}{2N} \right)^s ds
\]

(13)

where \( C \) is a contour in the complex \( s \)-plane separating the poles of \( \Gamma(N+s) \) located at \( s = -n - N \) from those of \( \Gamma(-s) \) located at \( s = n \), where \( n \) is any nonnegative integer. Substituting (13) into (12), and interchanging the order of integration yields

\[
I(a \mid r, \sigma_0, N) = \frac{1}{2\pi i} \int_C \Gamma(N+s)\Gamma(-s) \left( \frac{\sigma_0^2 \rho^2}{2N} \right)^s ds
\]

\[
\times \left[ \int_0^{\infty} J_0(\rho a)J_1(\rho r) \rho^{2s} d\rho \right] \left( \frac{\sigma_0^2 \rho^2}{2N} \right)^s ds.
\]

(14)

Several closed-form expressions can be derived for the bracketed factor in (14), each dependent upon the relative values of \( a \) and \( r \). Each case is discussed below.

Case 1 \( a < r \).

Here the bracketed term in the integrand of (14) is given by ([15, p. 692, eq. 6.574.1])

\[
\int_0^{\infty} J_0(\rho a)J_1(\rho r) \rho^{2s} d\rho
\]

\[
= \frac{1}{r} \frac{\Gamma(1+s)}{\Gamma(1-s)} \left( \frac{2}{r} \right)^{2s} 2F_1(1+s, 1; \frac{a^2}{r^2})
\]

(15)

where \( 2F_1 \) denotes the hypergeometric function. For our purpose, (15) is valid only for \( \max(-N,-1) < \text{Re}(s) < 1/2 \). Using the identity

\[
\frac{\Gamma(-s)}{\Gamma(1-s)} = -\frac{1}{s}
\]

(16)

and substituting (15) into (14), yields

\[
I(a \mid r, \sigma_0, N) = \frac{-1}{2\pi i} \int_C \frac{\Gamma(N+s)}{s} \left( \frac{2\sigma_0^2}{N^2} \right)^s ds
\]

\[
\times 2F_1(1+s, 1; \frac{a^2}{r^2}) ds.
\]

(17)

This integral can be evaluated by residue theory. To do this, we note that the function \( \Gamma(N+s) \) has
first-order poles located at $s = -n - N$ whose residues are $(-1)^n/n!$ for $n = 0, 1, 2, \ldots$, respectively. Coupled with the restriction that the contour $C$ must be contained in the vertical strip defined by $\max(-N, -1) < \text{Re}\{s\} < 1/2$, an appropriate closed contour can be found, as is shown in Fig. 2. It is composed of a vertical portion, $L_\rho$, and a semicircular portion $C_\rho$, whose radius is $\rho$. The union of $L_\rho$ and $C_\rho$ converges to the contour $C$ as $\rho$ becomes large. Since $2\sigma_0^2/Nr^2 > 0$, it can be shown that the integral along $C_\rho$ decays to zero as $\rho \to \infty$. (See Appendix A.) Therefore, we can neglect the placement of a portion of the contour outside the strip $\max(-N, -1) < \text{Re}\{s\} < 1/2$, since in the limit as $\rho \to \infty$ only the portion inside the strip contributes to the integration. This contribution is precisely $I(a \mid r, \sigma_0, N)$. With this justification, the integral in (17) is given by

$$I(a \mid r, \sigma_0, N) = \frac{(Nr^2/2\sigma_0^2)^N}{r \Gamma(N)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+N)} \left(\frac{Nr^2}{2\sigma_0^2}\right)^n \times \, _2F_1(1-n-N, -n-N; 1; a^2/r^2).$$ (18)

It is shown in Appendix B that this series, and those derived for the remaining two cases, are convergent.

**Case 2** $a > r$.

This case is handled similarly to Case 1, the only difference is the value of the bracketed term in (14) which is given by (115, p. 692, eq. 6.574.3)

$$\int_0^\infty I_0(\rho r)I_1(\rho r)\rho^{2s} \, d\rho = \frac{r}{a^2} \frac{\Gamma(1+s)}{\Gamma(-s)} \left(\frac{2}{a}\right)^{2s} \, _2F_1(1+s, 1+s; 2; r^2/a^2).$$ (19)
approaches a step function. This limiting case is easy to evaluate, for if the target strength goes to zero ($\sigma_0 \to 0$), then ([15, p. 667, eq. 6.512.3])

$$I(a \mid r, 0, N) = \int_0^\infty J_0(p r a) I_1(p r) p \, dp$$

$$= \begin{cases} 1/r & \text{for } a < r, \\ 1/2r & \text{for } a = r, \\ 0 & \text{for } a > r. \end{cases} \quad (25)$$

Substituting (25) into (10) yields $F_{r, F, C}(r) = F_r(r) c$, a reasonable result, since only clutter is presented to the receiver.

Finally, it should be noted that the confluent hypergeometric and hypergeometric functions have the series representations

$$\text{$_1F_1(a, b; x)$} = \sum_{m=0}^{\infty} \frac{(a)_m x^m}{(b)_m m!} \quad (26)$$

$$\text{$_2F_1(a, b; c; x)$} = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{(c)_m m!} \quad (27)$$

where

$$\cases{(a)_m = a(a + 1) \cdots (a + m - 1), \\
(a)_0 = 1} \quad (28)$$

is Pochhammer’s symbol. If either $a$ or $b$ is equal to $-M$, where $M$ is a nonnegative integer, then both series in (26) and (27) terminate at the $M$th term. Hence, if $N$ is an integer, then (18) and (21) are each composed of a series of polynomials of ascending order which simplifies the computation of $I(a \mid r, \sigma_0, N)$. More complicated algorithms must be used to calculate the hypergeometric functions if $N$ is not an integer; however, they are available in software packages developed to calculate special functions [17].

IV. EVALUATION OF THE KERNEL

In the previous section we derived alternate integral expressions for the kernel $I(a \mid r, \sigma_0, N)$ given by (17), (20), and (23). These integrals are more amenable to numerical integration since their integrands are not nearly as oscillatory as the integrand of (12), and decay at the same rate regardless of the values of $r$, $\sigma_0$, or $N$.

Furthermore, we need not numerically integrate these alternate expressions in their present form. First, one can verify that their integrands are conjugate symmetric about the imaginary axis.

It is convenient to consider a path of integration that is a vertical line in the complex plane such that $\text{Im}(s) < 0$. Thus, for $a < r$, with $s = x + jy$ (x is fixed), the integral in (17) is well approximated by

$$I(a \mid r, \sigma_0, N) \approx -\frac{(2\sigma_0^2/N r^2)^x}{\pi r \Gamma(N)} \int_0^\alpha \text{Re} \left\{ \frac{\Gamma(N + x + jy)}{x + jy} \right\} dy$$

$$\times F_1(1 + x + jy, 1 + x + jy; 1; a^2/r^2 e^{i y \ln(2\sigma_0^2/N r^2)}) \quad (29)$$

provided $\alpha$ is sufficiently large. In a similar fashion, for $a > r$ one finds

$$I(a \mid r, \sigma_0, N) \approx \frac{r(2\sigma_0^2/N a^2)^x}{\pi a^2 \Gamma(N)} \int_0^\alpha \text{Re} \left\{ \frac{\Gamma(N + x + jy)}{x + jy} \right\} dy$$

$$\times F_1(1 + x + jy, 1 + x + jy; 2; a^2/r^2 e^{i y \ln(2\sigma_0^2/N a^2)}) \quad (30)$$

and for $a = r$,

$$I(r \mid r, \sigma_0, N) \approx \frac{(2\sigma_0^2/N a^2)^x}{\pi r \Gamma(N)} \int_0^\alpha \text{Re} \left\{ \frac{\Gamma(N + x + jy) \Gamma(-2(x + jy))}{\Gamma^2(1 - x - jy)} \right\} e^{i y \ln(2\sigma_0^2/N r^2)} dy. \quad (31)$$

Second, portions of the integrands, the factors involving the gamma functions, can be precomputed. This can reduce the number of computations if the same abscissa points are used each time a numerical integration is performed.

Use of the alternating series presented in the previous section implies that a controlled numerical evaluation of $I(a \mid r, \sigma_0, N)$ is possible, and can imply that only a single integration need be performed to calculate detection probabilities. Certainly if the alternating series are used, then $I(a \mid r, \sigma_0, N)$ can be approximated by a truncated series. A bound for the error of this method is easy to determine: it is simply equal to the absolute value of the magnitude of the first term eliminated from the series.

V. NUMERICAL EXAMPLES

A. Weibull Clutter in Radar

Assume the amplitude of the matched-filter response to clutter follows the Weibull density function, i.e.,

$$f_{R,C}(r) = \frac{c}{b^c} r^{c-1} e^{-r/c/b^{c}} \quad (32)$$

where $b$ is the scale parameter, and $c > 0$ is the shape parameter related to the skewness of the density function. The Weibull density function is used extensively in failure-rate analysis [18]; however, it has found favor in radar detection theory since
it provides a good model for the matched-filter amplitude response to non-Gaussian clutter. This density function contains more area under its “tail” than does the Rayleigh density function \( (c = 2) \) which models the matched-filter amplitude response to Gaussian clutter.

Given (32), the expected power of the clutter is
\[
E \{ r_C^2 \} = \frac{C}{b^2} \int_0^\infty r^{\gamma+1} e^{-r/b^\gamma} \, dr = b^2 \Gamma (1 + 2/c).
\]

(33)

Without loss of generality we set \( b = 1 \). Thus, using (8), the signal-to-clutter-ratio (SCR) is
\[
\text{SCR} = \frac{2\sigma_0^2}{\Gamma (1 + 2/c)}.
\]

(34)

Furthermore, the detection threshold \( \eta \) can be derived from the cumulative probability function using the following equation:
\[
P_{fa} = \int_\eta^\infty f_{R,C}(r) \, dr = 1 - F_{R,C}(\eta) = e^{c\eta}.
\]

(35)

Thus, solving for \( \eta \) in terms of the probability of false alarm \( P_{fa} \) yields
\[
\eta = [-\ln(P_{fa})]^{1/c}.
\]

(36)

When possible, the series representations of the kernel were used with the truncation error set to \( 10^{-6} \). However, if \( Nr^2/2\sigma^2 > 4 \) in (18) and (24), or \( Ne^2/2\sigma^2 > 4 \) in (21), then numerical integration of the approximations to \( I(a \mid r, \sigma_0, N) \) given in (29), (30), and (31) was performed using a sixteen-point Gaussian quadrature with \( \alpha = 4 \), and \( x = -N/2 \) for \( N < 1 \), or \( x = -1/2 \) for \( N \geq 1 \). The choice of \( \alpha \) was found through experimentation. For \( \alpha > 4 \) the values yielded by the numerical integration of (29), (30), and (31) were consistent to five or more decimal places with those obtained using \( \alpha = 4 \).

Numerical integration of the product of the kernel with the clutter density function as given in (10) was accomplished using a five-point Gaussian quadrature applied over several concatenated intervals, and was based on the assumption that
\[
F_{R,T+C}(\eta) \approx \int_0^\beta f_{R,C}(a) I(a \mid \eta, \sigma_0, N) \, da
\]

(37)

for a sufficiently large value of \( \beta \). It was found that for \( N < 1 \), the kernel \( I(a \mid r, \sigma_0, N) \) had a decreasing jump discontinuity at \( a = \eta \). Thus, the numerical integration of (37) was performed using seven quadrature intervals: four equal-length quadrature intervals in the interval \([0, \eta]\), followed by three equal-length quadrature intervals in the interval \([\eta, \beta]\). The choice of \( \beta \) was based on the value of \( c \). For \( c < 1 \), the clutter amplitude density function decayed slowly, and a good choice for \( \beta \) was \( 1.5\eta \). Furthermore, for the first quadrature interval, the

local change of variable \( dy = ca^{-1}da \) was used to remove the integrable singularity of the clutter amplitude density function at \( a = 0 \). For \( c > 1 \), the clutter amplitude density function decayed rapidly, and a good choice for \( \beta \) was \( \eta + 5 \). These choices for \( \beta \) and the number of quadrature intervals were found experimentally; using a larger \( \beta \), or more quadrature intervals, did not cause a change in the first five decimal places of the detection probabilities.

All the numerical examples were performed on a NeXT\textsuperscript{TM} computer running Mathematica\textsuperscript{TM}, a software package designed to do symbolic mathematics, as well as numerical computations [19]. Mathematica\textsuperscript{TM} is capable of performing calculations to an almost arbitrary number of decimal places; however, the numerical accuracy for all internal computation was set to sixteen decimal places, the machine precision of the NeXT\textsuperscript{TM}. This is equivalent to double precision on most computers.

Defining the probability of detection as
\[
P_d = \int_\eta^\infty f_{R,T+C}(r) \, dr = 1 - F_{R,T+C}(\eta)
\]

(38)

receiver operator characteristic (ROC) curves were calculated for \( N \) equal to 0.25, 0.5, 1.0, 1.5, 3.0, and 6.0, \( c \) equal to 0.5, 1.0, and 2.0, and \( P_{fa} \) equal to \( 10^{-3} \) and \( 10^{-5} \) for the SCR spanning a 35 dB range. The curves are shown in Figs. 4—9. For \( c = 0.5 \), the ROC curves are “broad,” and show that even for a high SCR (25 dB or greater) the probability of detection can be marginal \( (P_d \approx 0.5) \). For \( c = 0.5 \), the clutter amplitude density function is also broad, implying that the contribution to the matched-filter response by the clutter component can, with high probability, exhibit a wide range of values. This, in turn, implies that the matched-filter output can fluctuate enough to have a significant chance of not exceeding the detection threshold for high values of the SCR.

B. Sea Surface Clutter in Sonar

Detection in sonar systems is often quite difficult since the ocean is an extremely hostile acoustical environment. Clutter exists at the ocean surface due
to the rough surface and bubble plumes produced by breaking waves, in the volume due to fish schools, and at the bottom due to sea mounts and the inhomogeneous composition of the floor. Noise is also present due to distant ships and breaking waves. The resulting matched-filter response to these sources is often non-Gaussian.

The data presented here were acquired from a Critical Sea Test, Phase II sea surface scattering measurement conducted in Tufts Abyssal Plain near Alaska in March 1992 [20]. The acoustic source was a 15 element, vertical line array with a half-wave spacing of 1 m, and the receiver was a 52 element, horizontal line array with a half-wave spacing of 0.75 m. Both arrays were towed by the same ship with the source array center at a depth of 135 m, and the receiver array at a depth of 200 m located 900 m behind the ship. Beamforming was performed on both arrays so that the sea surface was examined at grazing angles between 3 and 9 deg. Although some noise was present, and probably has some effect on the statistics of the data, the environment was dominated by the clutter.

Fig. 10 shows the normalized histogram of the matched-filter output derived from processing 13467 returns originating from a 0.6 s linear frequency modulated (LFM) sweep from 850 to 1090 Hz. The histogram is composed of all the samples of the matched-filter output for each processed ping. Due to the sampling period being approximately half that of the signal resolution, and an approximate 50% overlap in the beamforming, the effective number of
From (39) and (43) it can be shown that breaking waves were present. It is likely that only one when the wind speed was approximately 15 m/s, and near the surface since the measurement was conducted density function. It is thought that the shape of the tail function, whereas its tail follows an exponential function, in Fig. 10. The peak is similar to a Rayleigh density function. A plot of this fitted density function is also shown or two bubble plumes were present in any one data point since the beamforming and pulse compression of the matched-filter implied that the surface was resolved to an area of approximately 1000 m².

A chi-square goodness-of-fit test was applied to the data set to test the hypothesis that the estimated density could have produced the observed data set. The test statistic was defined as

$$
\ell = N \sum_{i=1}^{M} \left( \frac{o_i - e_i}{e_i} \right)^2
$$

Fig. 11. ROC curves for detecting a chi-square fluctuating target in sea surface acoustic clutter with $P_{fa} = 0.001$.

Integration yields

$$
\varphi(r) = \begin{cases} 
0 & r < r_1, \\
((r - r_1)/b_1)^\gamma & r_1 \leq r < r_0, \\
((r - r_2)/b_2)^\gamma + K & r_0 \leq r < \infty,
\end{cases}
$$

The cumulative density function is given by

$$
F_{R,C}(r) = 1 - e^{-\varphi(r)},
$$

From (39) and (43) it can be shown that

$$
\varphi'(r) = \frac{f_{R,C}(r)}{1 - F_{R,C}(r)}.
$$

This formula was used to estimate $\varphi'(r)$ using the normalized histogram as an estimate of $f_{R,C}(r)$. Estimates of the density function parameters were obtained graphically from a plot of the estimate of $\varphi'(r)$. They are

$$
r_1 = 0.001412 \\
b_1 = 0.00316228 \\
c_1 = 2.0 \\
r_2 = 0.004262 \\
b_2 = 0.00175439 \\
c_2 = 1.0 \\
K = 0 \\
r_0 = 0.004262.
$$

A plot of this fitted density function is also shown in Fig. 10. The peak is similar to a Rayleigh density function, whereas its tail follows an exponential density function. It is thought that the shape of the tail is due to occasional large bubble plumes that existed near the surface since the measurement was conducted when the wind speed was approximately 15 m/s, and breaking waves were present. It is likely that only one
VI. SUMMARY AND CONCLUSIONS

We have presented a method for calculating the probability of detecting a chi-square fluctuating target in a clutter-limited environment where the amplitude probability density function of the matched-filter response to clutter is arbitrary. Numerical examples were performed to demonstrate the utility of the method; one using a common radar clutter model, and the other using experimental sonar data. The resulting ROC curves show that \( P_d \) is highly dependent upon the fluctuation models of both the target and clutter. In particular, the ROC curves for \( N = 1 \) and \( c = 2 \) shown in Figs. 8 and 9 are those arising from the classical theory—that of a Rayleigh fluctuating target in Gaussian noise. The difference between these two curves and those shown in the remaining figures demonstrates how much the value of \( P_d \) can deviate from that predicted by the classical theory if the target or clutter follows a non-Gaussian fluctuation model.

We have two comments regarding the practical use of the methods presented in this work. First, detection probabilities are most accurately and efficiently calculated when the series expressions for \( I(a | r, \sigma_0, N) \) are used, since these offer the best opportunity for controlling error. This usually occurred for moderate-to-high SCR where \( P_d \geq 0.3 \). For small values of the SCR, \( P_d \) approaches 0, and numerical integration is required to calculate \( I(a | r, \sigma_0, N) \). This, however, should not be viewed as a shortcoming, since the most valuable portions of the ROC curve are for a moderate SCR where the performance of a detection system may become marginal, and consequently are of great interest to the system designer. Second, the methods presented here may work well for other target fluctuation models, and may lead to convenient closed-form expressions for \( I(a | r, \sigma_0, N) \).

APPENDIX A. VALIDITY OF THE CONTOUR INTEGRATION

In this Appendix, we show that if the contour of integration for the integral in (17) is extended to include a semicircular contour \( C_\rho \) in the left-hand complex half-plane, then the contribution to the integral along \( C_\rho \) approaches zero as the radius of the contour becomes infinitely large. To do this, we make use of the following asymptotic theorems for \( \rho \to \infty \):

\[
2F_1(\rho + 1, \rho + 3/2; 1/2; a^2/r^2) \sim \frac{(1 - a/r)^{2(\rho + 1)}}{2}
\]

and for any complex number \( N \), and \( \theta \neq \pi \),

\[
|\Gamma(N + \rho e^{i\theta})| \sim \sqrt{2\pi} e^{-N - \rho \cos(\theta) + (N + 1/2 + \rho \cos(\theta)) \ln \rho - \rho \theta \sin(\theta)}.
\]

We do not prove these theorems here; however, (48) follows from the triangle inequality as applied to the series representation for \( 2F_1(s + 1, s; 1; a^2/r^2) \), and a common property of the hypergeometric function ([16, p. 556, eq. 15.1.9]), and (49) follows from Stirling’s asymptotic approximation.

From Fig. 2, we see the contour \( C_\rho \) can be constructed using a radial sweep from \( \theta = \pi/2 + \epsilon \), to \( \theta = 3\pi/2 - \epsilon \) for some \( \epsilon > 0 \). We avoid the poles of \( \Gamma(N + s) \) if we choose \( \rho \) to be any noninteger real number. Thus, we consider the limit of integral along \( C_\rho \) as \( \rho \to \infty \) by letting \( \rho = \rho_n \), where \( \{\rho_n\}_{n=1}^\infty \) is a monotonically increasing sequence of positive noninteger values. Furthermore, it is sufficient to consider the integral along a semicircular path \( C_\rho^+ \) of radius \( \rho \) for \( \theta \in (\pi/2, \pi) \) since 1) it avoids integrating over the poles, 2) the integrand of (17) is conjugate symmetric about the real axis, and 3) equation (49) will be valid since \( \theta < \pi \). (We avoid \( \theta = \pi \) since it is only one point, is of measure zero, and can be eliminated from the integration.)

We define the integral

\[
J \doteq \int_{C_\rho^+} \frac{\Gamma(N + s)}{s} z^s 2F_1(1 + s, s; 1; a^2/r^2) \, ds
\]

where \( z \) is a nonnegative real number. Using the parameterization \( s = \rho_n e^{i\theta} \), we have

\[
|J| \leq \int_{\pi/2}^{\pi} \left| \Gamma(N + \rho_n e^{i\theta}) \right| \times z^{\rho_n \cos(\theta)} |2F_1(1 + \rho_n e^{i\theta}, 1 + \rho_n e^{i\theta}; 1; a^2/r^2)| \, d\theta.
\]

However, from (48) and (49) we see that the asymptotic value of the integrand of (51) is

\[
\sqrt{\frac{\pi}{2}} \exp\left(\frac{-N - \rho_n(1 - \ln(z \rho_n)) \cos(\theta) + \theta \sin(\theta) + 2\ln(1 - a/r) + (N - 1/2) \ln(\rho_n) - 2\ln(1 - a/r)}{a^2/r^2}\right).
\]
Therefore, the integrand of \( J \) goes to zero as \( \rho_n \to \infty \), implying the integral \( J \) goes to zero as \( \rho_n \to \infty \). This, in turn, implies that the use of a closed contour for the evaluation of (17) via residue theory is valid.

APPENDIX B. THE CONVERGENCE OF THE KERNEL SERIES REPRESENTATIONS

Consider the series

\[
S = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma^2(n+1) \left( \frac{N r^2}{2 \sigma_0^2} \right)^n = \sum_{n=0}^{\infty} a_n \tag{53}
\]

where \( N \) is any nonnegative real number. One can find an asymptotic expression for \( a_n \) by noting that \( \Gamma(n+1) = n! \), and using the following asymptotic properties of the gamma and exponential functions which hold for \( n \to \infty \):

\[
\Gamma(n) \sim \sqrt{2\pi} n^{n-1/2} e^{-n} n^{-n-1/2},
\]

\[
(n + \alpha)^n \sim n^n e^{\alpha}, \quad (n + \alpha)^m \sim n^m.
\]

Therefore,

\[
a_n \sim \frac{2^{2N-3/2}}{\pi n^2} \left( \frac{2Ne^2}{\sigma_0^2} \right)^n.
\]

Since the series in (53) is a sum of positive terms, we can test for its convergence using the limit comparison test. Consider the convergent series \( \sum_{n=1}^{\infty} 1/n^2 = \sum_{n=1}^{\infty} b_n \). The asymptotic value of the ratio of \( a_n \) and \( b_n \) is

\[
\frac{a_n}{b_n} \sim \frac{2^{2N-3/2}}{\pi} \left( \frac{2Ne^2}{\sigma_0^2} \right)^n.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0
\]

proving that the series in (53) is convergent (\( S < \infty \)).

Using the comparison test, and the series in (53), we now show that the series representations for \( I(a \mid r, \sigma_0, N) \) are absolutely convergent. Consider the case of \( a < r \). Given that \( \Gamma(x + 1) = x \Gamma(x) \), and the property

\[
2F_1(a,b;c;1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}
\]

where \( c \neq 0, -1, -2, \ldots \), and \( c > a + b \), we find that

\[
\left| 2F_1(1-n-N,-n-N;1;a^2/r^2) \right| \leq 2F_1(1-n-N,-n-N;1;1) = \frac{(n+N)\Gamma(2n+2N)}{\Gamma^2(n+N+1)}. \tag{58}
\]

Thus, given the series representation for \( I(a \mid r, \sigma_0, N) \) in (18), and the inequality in (59), it can be shown that

\[
|I(a \mid r, \sigma_0, N)| \leq \left( \frac{N r^2}{2 \sigma_0^2} \right)^N \frac{S}{\Gamma(N)} < \infty \tag{60}
\]

implying that (18) is an absolutely convergent alternating series. In a similar fashion, one can show that the series representations for \( I(a \mid r, \sigma_0, N) \) for \( a > r \) and \( a = r \) given in (21) and (24) are also absolutely convergent.

REFERENCES


Jao, K. J., and Elbaum, M. (1978)  
First-order statistics of a non-Rayleigh fading signal and its detection.  

Jao, K. J. (1984)  
Amplitude distribution of composite terrain radar clutter and the K-distribution.  

Gradsteyn, I. S., and Ryzhik, I. M. (1965)  
*Table of Integrals, Series, and Products.*  

Abramowitz, M., and Stegun, I. A. (1964)  
*Handbook of Mathematical Functions.*  

Baker, L. (1992)  
*C Mathematical Function Handbook.*  

Walpole, R. E., and Myers, R. H. (1978)  
*Probability and Statistics for Engineers and Scientists* (2nd ed.).  

*Mathematica: A System for Doing Mathematics by Computer* (2nd ed.).  

Intermediate-pulse direct-path measurements of surface scattering in CST-7 Phase II.  
In *Critical Sea Test 7, Phase II—Part I: Surface Scattering and Volume Measurements,*  

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