is proposed and evaluated. A GMSK modulation with $BT \approx 0.9$ is required to overcome both effects. This results in a rather modest spectrum shaping as compared with $BT = 0.3$, used in the European digital cellular telecommunications system (see Fig. 2). With $BT \approx 0.9$ the ISI is low and the degradation of a coherent MSK receiver is negligible. The MSK signal has a constant envelope and is therefore insensitive to nonlinear amplification in the transmitter, a property not achievable with bandpass filtering of an MSK signal. GMSK is not more difficult to implement than filtered MSK, but it has some unique characteristics which make it recommendable for those cases, where a spectral shaping of MSK is required.

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Requirements for Optimal Glint Reduction by Diversity Methods

Target "glint" (the difference between the phase-front gradient  
of the scattered field and the true target direction vector) is  
related to the analytic properties of the scattered field. The  
resulting model is used to reinterpret diversity methods for glint  
reduction and to more accurately specify system requirements.  
A relationship for estimating the best possible improvement in  
target bearing estimation is also developed.

1. INTRODUCTION

Interference effects in radar scattering from  
extended unresolved targets have been recognized  
and examined for five decades [1-4]. Despite these  
long and continuing efforts, the associated problem  
of accurate tracking of complex targets by small  
missile systems is still being studied and techniques for  
mitigating tracking errors are still being devised. The  
original work in the glint tracking problem was largely  
concerned with a statistical characterization of glint  
"noise"—the idea being to apply filtering techniques  
to eliminate it [4-6]. As a consequence, much of the  
subsequent relevant literature has been dominated by

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statistical considerations and the uninitiated reader may expend considerable effort in trying to understand the role of various target models and the conclusions which are drawn from them. Worse, it may be very difficult to understand just what glint is and to sort-out fact from conjecture.

In the following we present a nonstatistical model based on the analytic properties of fields scattered from finite targets. This model turns out to be very simple and explains all of the behavior of glint in an easy-to-interpret way. Moreover, the model is quite general and independent of target details (depending only upon target dimensions). From this model we are able to construct general criteria for designing effective aspect-diverse or frequency-diverse radar tracking systems.

Beginning with Section II, the problem is presented in its mathematical form, the notation is developed and the scattering model is defined. Sections III and IV examine the consequences of this model as it relates to aspect and frequency diversity glint mitigation methods and show that the frequently quoted requirements for effective glint reduction are not completely accurate. Also developed is an expression for the "optimal" accuracy of a monopulse system defined as the minimal distance from the target centroid attainable by an aspect or frequency-diverse system in the presence of glint error. Finally, the implications of this analysis on rms glint error reduction are examined.

II. RADAR GLINT

A radar launches an incident wave directed at a target and measures the resulting scattered wave from it. When the target is a simple point, the constant phase surfaces of the scattered field ("phase fronts") will consist of concentric spherical shells centered on the target. For tracking purposes, interferometric methods are used to determine phase-front normals and this information is used to estimate target bearing [7, 8]. The phase-front normals lie parallel to the phase gradient vector and the standard approach in examining pointing vector behavior is to study the components of the phase gradient.

Tracking glint error comes about when the target is not a simple point scatterer. Real targets can be considered to be made up of a collection of scattering elements with each element responding to the incident wave and each contributing to the scattered field. The simplest approximation treats these subscatterers as independent and noninteracting. In this "weak scatterer" treatment, the incident wave is the only wave that a scatterer sees and the secondary wave scattered from one point cannot excite another.

We fix a coordinate system to the target and consider a monostatic scattering situation in which the transmitter and receiver are colocated. The weak scatterer approximation far-field response \( E(k) \) due to a harmonic excitation of a target can be written as a linear superposition of waves with strength \( |\rho(r)| \) radiating from location \( r \):

\[
E(k) = \int_D \rho(r) \exp(i k \cdot r) dr. \tag{1}
\]

Here, \( D \) is the support of the local scatterer density function \( \rho(r) \) (scaled by range), \( k \) is the wave vector with \( |k| = 2\pi f/c = 2\pi/\lambda \) is the speed of wave propagation, \( f \) the wave frequency, and \( \lambda \) the wavelength. (We have suppressed the \( \exp(-i2\pi ft) \) time dependence.) This far-field approximation can be expected to be quite accurate up to the late end-game, at which point additional missile guidance corrections are usually treated as having only limited effectiveness. Moreover, since missile antennas are almost always mounted on moving gimbals so that the target can be "held" to lie within the near-linear region of the antenna pattern, it relatively easy to correct the response (1) for variations in the receive aperture. (Consequently, this aperture variation presents only a minor complication and is not considered in the following.)

Consider values of \( k \) restricted to lie within a plane containing both the target and the radar. If \( \theta \) denotes target aspect in this plane (see Fig. 1), then (1) can be written

\[
E(f, \theta) = \int_D \rho(x, y) \exp(i(2\pi f/c)(x \sin \theta - y \cos \theta)) dx dy
\]

where now \( \rho(x, y) \) represents the source function integrated along the direction orthogonal to the \( x-y \) plane. Denote \( \xi \equiv (2\pi f/c) \sin \theta \) and \( \psi \equiv -(2\pi f/c) \cos \theta \). In the usual small angle approximation, \( \xi \approx 2\pi f \theta/c \) and \( \psi \approx -2\pi f/c \), and we can write (2) as the iterated integral

\[
E_f(\xi) = \int_{\mathbb{R}} \tilde{\rho}_f(x) \exp(i \xi x) dx \tag{3}
\]

where \( \tilde{\rho}_f(x) = \int \rho(x, y) \exp(-i2\pi f y/c) dy \) and each integral has limits appropriate to the target support. (In particular, \( \tilde{\rho}_f(a), \tilde{\rho}_f(b) \neq 0 \).)

Equation (3) can be used to continue \( E_f(\xi) \) into the complex \( w \)-plane so that

\[
E_f(w) = \int_a^b \tilde{\rho}_f(x) \exp(iwx) dx. \tag{4}
\]

If the support interval \( (a, b) \) is finite then \( E_f(w) \) is an entire function of exponential type and can be represented by the Weierstrass canonical product:

\[
E_f(w) = E_f(0) e^{P(w)} \prod_{n=1}^{\infty} \left( 1 - \frac{w}{w_n} \right) \exp(w/w_n) \tag{5}
\]

where \( P(w) \) is a polynomial, the product is over all complex zeros \( \{w_n\} \) of \( E_f(w) \) and we have assumed that \( E_f(0) \neq 0 \).
Titchmarsh [9] has shown that when $E_f(w)$ is of the form of (4), then the general form of (5) can be significantly simplified. On the real-$w$ axis we can write

$$E_f(\xi) = E_f(0) \exp \left( i \frac{a + b}{2} \xi \right) \prod_{n=1}^{\infty} \left( 1 - \frac{\xi}{w_n} \right). \quad (6)$$

This product represents the scattered field as a function of the real variable $5 = 2rf_e/c$ in factors of the complex zeros of the field continued into the complex plane. (We return to this important point later when we discuss the relationship between “glint spikes” and “amplitude fades”.)

The quadrature phase $\varphi_f(\xi)$ of the scattered field is defined by

$$E_f(\xi) = |E_f(\xi)| \exp(i \varphi_f(\xi)).$$

The derivative of this phase can be found from

$$\frac{c}{2\pi f} \frac{\partial \varphi_f(\xi)}{\partial \theta} = \frac{c}{2\pi f} \text{Im} \left( \frac{E_f(\xi)^2}{E_f(\xi)} \frac{\partial E_f(\xi)}{\partial \theta} \right). \quad (7)$$

Applying (7) to (6) yields

$$\varphi_f(\xi) = \frac{c}{2\pi f} \int \left( \frac{1}{2} \text{Re}(w_n) - \text{Im}(w_n) \right) \frac{2\pi f(\theta - \text{Re}(w_n)) + \text{Im}(w_n)^2}{\text{Im}(w_n)^2 - \text{Re}(w_n)^2} \, d\theta. \quad (8)$$

This is the glint error in units of linear cross-range coordinate at the target.

In developing this result, we have assumed that the location of the zeros $\{w_n\}$ does not depend upon target aspect. However, since different subcomponents of a complex target are excited by the incident field at different target orientations, there is usually a dependence of zero location on $\theta$. Consequently, (8) must be viewed as an approximation which is valid only over a small range of aspect angles for which the locations of the zeros can be taken as fixed.

Equation (8) is proportional to the pointing distance away from the origin of the coordinate system fixed within the target and displays all of the behavior associated with glint (for example, it displays the strong correlation between glint spikes and amplitude fades [10]). When $(a + b)$ is a slowly varying function of aspect the second term is vanishingly small. The first term does not vary with small changes in aspect and $\frac{1}{2}(a + b)$ is known as the cross-range tracking “centroid” of the target. As we have discussed, larger aspect variations are often accompanied by changes in the value of $(a + b)$ and so the centroid varies in position (this is known as “bright-spot wander”). If the change in centroid position is rapid enough, then the second term in (8) can take on large transient values and we may observe associated spike-like behavior.

Note that since this spike-like behavior is associated with a rapid change in $(a + b)$, it generally separates the domain of $\frac{\partial \varphi_f(\theta)}{\partial \theta}$ into subdomains over which (8) is valid to within the small angle approximation: in each subdomain, $a, b$, and $\{w_n\}$ can be assumed to be approximately fixed.

Within each of these subdomains there is also a regular spike-like behavior determined by the third term of (8). The $\{w_n\}$ are defined as the points on the complex $w$-plane at which $E_f(w)$ vanishes. If we write $w_n = \text{Re}(w_n) + i \text{Im}(w_n)$ then

$$\text{Im} \left( \sum_{n=1}^{\infty} \frac{1}{2\pi f(\theta - \text{Re}(w_n))^2 + \text{Im}(w_n)^2} \right) = \sum_{n=1}^{\infty} \frac{\text{Im}(w_n)}{2\pi f(\theta - \text{Re}(w_n))^2 + \text{Im}(w_n)^2} \quad (9)$$

and we can see that a zero $w_n$ makes a significant contribution to the sum in (9) only if its imaginary part is sufficiently small.

Let $w_1 = |w_1| \exp(i \gamma_1), w_2 = |w_2| \exp(i \gamma_2), \ldots$, be the zeros arranged in order of increasing modulus. When $\tilde{\varphi}_f$ is defined as in (3) with $a \neq b$, the asymptotic form of the zeros is given by [11]

$$w_n = \frac{n \pi}{b - a} + i \frac{1}{2(b - a)} \ln \left( \frac{\tilde{\varphi}_f(a)}{\tilde{\varphi}_f(b)} \right) + \epsilon_n. \quad (10)$$

where $\epsilon_n \to 0$ as $n \to \infty$. (If $a = b$ then the third term of (8) vanishes.) $E_f(w)$ is known as a “sine-type” function and has zeros that lie (approximately) on a regular lattice determined by the support of $\tilde{\varphi}_f$. When we write $\tilde{\varphi}_f(x) = |\tilde{\varphi}_f(x)| \exp(i \gamma_f(x))$ then (10) becomes

$$w_n = \frac{n \pi}{b - a} + i \frac{1}{2(b - a)} \ln \left( \frac{\tilde{\varphi}_f(a)}{\tilde{\varphi}_f(b)} \right) + \epsilon_n. \quad (11)$$

Equations (9) and (11) allow us to make a qualitative description of glint. When $2\pi f(\theta - c)$ $= \text{Re}(w_n), n = 1, \ldots, \infty$, the third term of (8) adds an additional angular glint error of approximately

$$\theta_{\text{spike}} \approx \frac{2(b - a)}{\ln \left( \frac{\tilde{\varphi}_f(a)}{\tilde{\varphi}_f(b)} \right)}. \quad (12)$$
This approximation assumes that the spike is large. In practice, of course, there is a finite number of such glint spikes within each subdomain of “fixed” $a$ and $b$. However, the density of these zeros increases with increasing frequency. Note that the size of the error spike is determined by the behavior of $\tilde{\beta}_f$ near the end point of the interval $(a, b)$ and that when $|\tilde{\beta}_f(a)/\tilde{\beta}_f(b)| \approx 1$ these spikes can become quite large.

These results are illustrated in Fig. 2 which displays the calculated glint error for a numerically synthesized target. This target consists of 10 point scatterers with fixed locations on a plane (Fig. 3). The support of this target had cross-range dimension of 2 m and down-range dimension of 5 m. The simulated frequency was 10 GHz. Since the subscatterers are simple points, an easy calculation reveals that $\tilde{\beta}_f(x) = \sum_{n=1}^{10} A_n \exp(-i2\pi f y_n/c)\delta(x-x_n)$, where $A_n$ is the strength of the $n$th scatterer. The $\{A_n\}$ were randomly chosen to lie within $0.5 < A_n < 1$, with the exception that the cross-range endpoints were selected to obey $A_a/A_b = 2$. This selection of parameters allows for the determination of the overall behavior of the glint spikes from (12) and (13) (below).

Of course, this model applies to more general targets. Fig. 4 is a plot of the phase derivative measured from a target rotating on a fixed pedestal [12]. We have chosen a region that displays the effects of all three terms of (8). At aspects less than about $9^\circ$, the error is a combination of the constant first term and the spikes of the third term whose spacing is determined by some value of the (small angle) cross-range extent $(b - a)$. At an aspect of about $1^\circ$, the effective cross-range extent undergoes an abrupt shift and the error is dominated by the second term of (9). For aspects greater than this, the target centroid (“constant” first term) has shifted and the regularity of the glint spikes in this new region has also changed slightly.

III. DIVERSITY TECHNIQUES FOR GLINT MITIGATION

Because the first term in (8) represents the target centroid, its effect on tracking error should not be expected to be very significant. However, the errors due to the periodic spikes of the third term of (8) and the spurious spikes due to the second term can be quite large. Glint mitigation techniques invariably attempt to reduce the effects of these spikes and so we examine (9).

By construction, there is no zero at $f\theta = 0$. If we fix $\theta = \theta_0$ (where $\theta_0$ is “small”), then we can conclude from equation (11) that $f_n \propto n$, $(n = 1, \ldots, \infty)$ are the frequencies for which the denominator of (9) attains a minimum. Consequently, not every $w_n$ causes a glint spike: only those zeros whose indices correspond to $2\pi f_n \theta_0/c = \text{Re}(w_n)$ or, similarly, $2\pi f_n \theta_0/c = \text{Re}(w_n)$ will make a major contribution to glint error. When $n$ is small (say $n < N$) the associated zeros $w_n$ are determined by the behavior of the function $\tilde{\beta}_f$ within

Fig. 3. Location of scatterers used in calculating glint error of Fig. 2. Scatterer strength proportional to spot diameter (inset).
the support \((a,b)\) and this behavior is very sensitive to the overall structure of the target being examined [11]. These terms become important when \(f\) is small. However, when \(n > N\) the zeros are determined by the behavior of \(\hat{\rho}_{f}\) only at the endpoints of the target support. The number \(N\) can be considered to denote the number of "degrees of freedom" associated with the target: \(N\) will be large for complex targets and small for simple ones. We are interested in the case when \(f\) is large. "High-frequency" glint is generally defined by the condition \((b - a)/\lambda \gg 1\) or \((y_b - y_a)/\lambda \gg 1\) \((y_a\text{ and } y_b\text{ are defined as in Fig. 1})\). For these high frequencies, we may assume that \(n > N\) and take (11), with \(\epsilon_n > N\) small, to be a good approximation. (The notion of "high frequency" and the exact relationship between \(f_n\) and \(n\) is examined below.)

At high frequencies, the local glint spikes lie on a regular lattice with spacing

\[
\Delta \approx \frac{c}{2(b-a)}. \tag{13}
\]

This lattice is superimposed on \(f\theta\)-space and, by varying either frequency or target aspect, we sample the field in the region surrounding the spike.

**Space or Aspect Diversity Glint Reduction:** When \(f\) is fixed we can sample \(f\theta\)-space only by altering the aspect \(\theta\) from which the data are collected. If \(2\pi f\theta/c = \text{Re}(\omega_n)\) corresponds to a glint spike then, by altering \(\theta\) to \(\theta + \delta \theta\), where

\[
\delta \theta \approx \frac{1}{2} \left( \frac{\Delta}{f} \right) = \frac{1}{2} \left( \frac{c}{2f(b-a)} \right) = \frac{1}{2} \left( \frac{\lambda}{2(b-a)} \right),
\]

we effectively sample the field at a "glint minimum". (Note that the \(\delta \theta\) given by (14) is simply half the common "lobe width" in angle space.) In practice, of course, we do not generally know if we are on a spike and must sample over a range of aspects measuring \(\approx 2\delta \theta\). This sampling can be performed by physically extending the aperture of the radar system [13] or, as in inverse synthetic aperture radar (ISAR) imagery, by allowing the target to maneuver over time [14].

**Frequency Diversity Glint Reduction** (cf., [13, 15-19]): When \(f\) is allowed to vary the situation is somewhat different. Now we presume that \(\delta \theta\) is fixed and use the variation of \(\hat{\rho}_f(x)\) with \(f\) to move away from the spike.

For large \(f\), we can asymptotically evaluate \(\hat{\rho}_f(a)\) and \(\hat{\rho}_f(b)\) and conclude [20]

\[
\rho_f(a) - \rho_f(b) \approx 2\pi f(y_a - y_b)/c. \tag{15}
\]

If we vary \(f\) to \(f + \delta f\) in order to move away from a spike at \(f\theta\), then (11) implies that the minimum required bandwidth obey

\[
\delta f \approx \frac{1}{2} \left( \frac{\Delta}{\delta \theta} \right) = \frac{1}{2} \left( \frac{f}{n + (y_b - y_a)/\lambda} \right). \tag{16}
\]

This result is different in form from (14) in that it depends on the index \(n\) \((n > N)\) of the spike in question. When the downrange extent of the target is large compared with the radar wavelength (specifically, when \((y_b - y_a)/\lambda \gg n\)) then

\[
\delta f \approx \frac{f}{2(y_b - y_a)}. \tag{17}
\]

This result cannot ignore internal target structure however, and \(\delta f\) will not become infinitely large as \((y_b - y_a) \rightarrow 0\). When \((y_b - y_a)/\lambda \ll n\) (16) becomes

\[
\delta f \approx \frac{f}{2n}. \tag{18}
\]

**IV. DISCUSSION**

Since standard glint mitigation deals only with the third (spiky) term of (8) we consider the improvement due to the reduction technique in comparison with the magnitude of the spike. Let \(g_{\text{spike}}\) denote the value of the glint error when \(2\pi f\theta/c = \text{Re}(\omega_n)\) and \(g_{\text{min}}\) denote the magnitude of the remaining error after reduction. Since the spacing of zeros is uniform, then \(g_{\text{min}}\) corresponds to values of \(2\pi f\theta/c\) that lie half-way between the zeros. According to the results of Section III, these values will be guaranteed by sufficient sampling over \(2\delta \theta\) in aspect or over \(2\delta f\) in bandwidth. The optimal reduction obtainable can be determined from (9) by assuming that the spikes are sufficiently large and isolated that each can be treated separately.

We obtain

\[
g_{\text{min}} \approx \frac{1}{g_{\text{spike}}} \left[ 1 + \left( \frac{\pi}{2} \right) g_{\text{spike}}/(b-a) \right]^2. \tag{19}
\]

When \(g_{\text{spike}}/(b-a) \gg 1\) we achieve significant improvement and (19) simplifies to

\[
g_{\text{min}} \approx \frac{g_{\text{spike}}}{g_{\text{spike}}} \left( \frac{2(b-a)}{\pi} \right)^2. \tag{20}
\]

However, when \(g_{\text{spike}} \approx (b-a)\) the correction is not as significant and, when added to the centroid term of (8), the tracking point may lie significantly away from the target centroid.

The glint from this inter-spike region does not generally vanish and this fact means that we can never completely eliminate radar tracking error by accounting only for the spikes. From (12) and (19) we have

\[
g_{\text{min}} \approx \frac{2(b-a)\ln(r)}{\pi^2 + \ln^2(r)}, \tag{21}
\]

where \(r \equiv |\hat{\rho}(a)/\hat{\rho}(b)|\). This residual error vanishes when \((b-a) = 0\) or when \(r = 0, \infty\) (scatter strength at \(a\) is infinitely weaker or stronger that the strength at \(b\)) or \(r = 1\) (equal scatterer strength). When \(r = \exp(\pm \pi)\) the residual takes on its maximum value of \(g_{\text{min}} = (b-a)/\pi\).
The question of the rms glint reduction that can be expected from an implementation of these ideas is a difficult one because it may not be possible to achieve \( g_{\text{rms}} \) in a real system and we have not discussed ways to go about attempting this. Often, a weighted average of \( g \) over the frequency band or over the range of aspects is used to estimate \( g_{\text{rms}} \). The problem then becomes one of selecting the optimal weighting scheme [13-19]. Assuming that we are able to determine \( g_{\text{rms}} \) using some diversity algorithm, then we can estimate the rms reduction by integrating over one lobe in frequency (or angle) space.

For uncorrected glint error,

\[
g_{\text{rms}}^2 \approx \frac{1}{2\pi} \int_{-\delta f}^{\delta f} \left( g(f) - \frac{a + b}{2} \right)^2 \, df.
\]  

(22)

Assuming the lobes are sufficiently distinct, we can use a simple term of (9) to evaluate this integral and obtain

\[
g_{\text{rms}}^2 \approx 2(b - a)^2 \left( \frac{1}{\pi^2 + \ln^2(r)} + \frac{\arctan(\sqrt{r}/\ln(r))}{\pi \ln(r)} \right).
\]

(23)

Note that \( g_{\text{rms}} \) vanishes when \( r = 0, \infty \) or when \( (b - a) = 0 \). (Of course, this is somewhat artificial since the case \( r = 0, \infty \) is precluded by construction.) Combining this result with that of (21) yields

\[
g_{\text{rms}}^2 \approx 2\pi \ln^2(r) + \pi \ln^3(r) + (\pi^2 + \ln^2(r))^2 \arctan(\sqrt{r}/\ln(r)).
\]

(24)

This ratio represents the optimal reduction that can be achieved.

As a simple example, consider the target defined by Fig. 3 and the associated glint error data of Fig. 2. Equation (23) estimates \( g_{\text{rms}} \approx 6 \text{ m}^2 \) while the value obtained by direct calculation is \( \approx 7 \text{ m}^2 \). Equation (24) yields \( g_{\text{rms}}^2/\theta_{\text{rms}}^2 \approx 0.012 \) which is the same value obtained from the data.

V. SUMMARY AND CONCLUSION

In the foregoing analysis we have developed a general model for glint error. This model relies on the weak scatterer far-field approximation but is otherwise independent of the detailed interaction between the incident field and the specific target. As a consequence, we have been able to establish global results that can be expected for common glint error, including a relationship for the greatest reduction that can be expected by using diversity methods.

This "optimal reduction" may be attainable in realistic systems and we have also developed general relationships for the required bandwidth (for frequency-diverse methods) and for the required aspect extent (in the case of space-diverse methods, although this is expected to be of less interest). Finally, a new model for glint has been examined and shown to qualitatively display all of the known properties of glint. Equation (8) provides a much simpler method for examining and modeling glint than the \( N \)-point or \( N \)-shape scattering models commonly in use. While we have not discussed algorithms for attaining optimal glint reduction, it might be possible to fit diversity data to this new model and thereby meet the goal of (24).

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a formula for inverting a square matrix partitioned in the form

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

where \(A\) is \(n \times n\), \(D\) is \(m \times m\), and \(B\) and \(C\) are conformably dimensioned. If \(A\) is nonsingular, then

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} + A^{-1}BEC & -A^{-1}BE \\
-C & E
\end{bmatrix}
\]

(1)

where \(E = (D - CA^{-1}B)^{-1}\), follows easily from the block LU factorization

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
CA^{-1} & I
\end{bmatrix} \begin{bmatrix}
A & B \\
0 & D - CA^{-1}B
\end{bmatrix}
\]

Thus

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} - A^{-1}BE & 0 \\
-C & E
\end{bmatrix}
\]

(2)

gives (1). The authors of [4] consider the special case in which \(B\) is a column vector, \(C\) is a row vector, and \(D\) is a scalar.

Bodewig [1] cites nine independent discoverers of the general formula (1), the earliest of which are Georg Frobenius (1849–1917) and Issai Schur (1875–1941). The result is sometimes referred to as the Frobenius-Schur formula. The names of Sherman, Morrison, and Woodbury [6, 8] are frequently cited in the statistics and numerical linear algebra literature for both the block matrix inverse problem as well as the closely related formula for the inverse of a sum of matrices

\[
(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}.
\]

An excellent survey of formulas for both (1) and (2), including a very thorough historical analysis (somewhat at variance with Bodewig [1]), is contained in a paper by Henderson and Searle [3]. Their bibliography is one of the most complete available on the subject, and they present a scholarly discussion of historical precedent for the formula (1).

History aside, however, the more serious deficiency in [4] is that the proposed procedure can have severe numerical difficulties. The most serious problem is that intermediate matrices may be quite ill conditioned with respect to inversion, even though the “full-sized” matrix is quite well conditioned. The numerical inaccuracies incurred in computing intermediate quantities may then render subsequent computed quantities essentially meaningless. A related difficulty is the phenomenon of so-called “growth.” Wilkinson gave an example [7, p. 327] of a family of matrices \(W_n\) (for \(n \geq 1\),

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Comment on “Inversion of all Principal Submatrices of a Matrix”

The paper by Koç and Chen [4] includes new formulas for computational complexity, but the method itself is already known and has dubious numerical stability. A poorly conditioned example is shown.

Koç and Chen [4] have rediscovered a special case of a matrix inversion formula that can be found in many sources on matrix methods (e.g., [2, 5]). This is