Correspondence

In [1], the optimal steady-state gains and associated error covariances for an $\alpha$, $\beta$, $\Gamma$ filter with a piecewise constant jerk as a maneuver model are presented. While the results given in [1] are correct, the gains and error covariances can be found in a much simpler manner. The simplified expressions for the gains and the error covariances are given, along with the code for a simple program that computes the gains for a given tracking index.

INTRODUCTION

In [1], the optimal steady-state gains and associated error covariances for an $\alpha$, $\beta$, $\Gamma$ filter with a piecewise constant jerk as a maneuver model are presented. While the results of [1] are correct, the steady-state gains and error covariances can be found in a much simpler manner that is similar to the methods given in [2 and 3]. The steady-state error covariance matrix is found to be a simple function of $\alpha$, $\beta$, $\Gamma$, and the sample period $T$. Three simultaneous equations in $\alpha$, $\beta$, and $\Gamma$ are presented for computing the gains when given a tracking index $\Upsilon$. A simple MATLAB [4] program which calculates the steady-state gains for a given tracking index is also presented. In order to maintain notation similar to that of [2 and 3], $\Upsilon$ will denote the tracking index, whereas in [1], $T_I$ denoted the tracking index. Note that $\Upsilon$ is used because the tracking index of this problem differs slightly from the those found in [2 and 3].

MATHEMATICAL MODEL

The dynamics model commonly assumed for a target in track is given by

$$X_{k+1} = F_k X_k + G_k w_k \tag{1}$$

where $w_k \sim N(0,Q_k)$ is the process noise and $F_k$ defines a linear constraint on the dynamics. The target state vector $X_k$ contains the position, velocity, and acceleration of the target at time $k$. The linear measurement model is given by

$$Y_k = H_k X_k + n_k \tag{2}$$

where $Y_k$ is usually the target position measurement and $n_k \sim N(0,R_k)$. The Kalman filtering equations

Manuscript received May 14, 1993.


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associated with the state model in (1) and the measurement model in (2) are given by the following equations [3].

**Time Update:**

\[
X_{k|k-1} = F_{k-1}X_{k-1|k-1} \quad (3)
\]
\[
P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k}^T + G_{k-1}Q_{k-1}G_{k-1}^T. \quad (4)
\]

**Measurement Update:**

\[
K_k = P_{k|k-1}H_k^T[H_kP_{k|k-1}H_k^T + R_k]^{-1} \quad (5)
\]
\[
X_{k|k} = X_{k|k-1} + K_k[H_kX_{k|k-1} - X_{k|k-1}] \quad (6)
\]
\[
P_{k|k} = [I - K_kH_k]P_{k|k-1} \quad (7)
\]

where \(X_{k|k}\) and \(P_{k|k}\) denote the mean and error covariance of the state estimate, respectively. The subscript notation \((k | j)\) denotes the state estimate for time \(k\) when given measurements through time \(j\), and \(K_k\) denotes the Kalman gain.

**THE \(\alpha, \beta, \Gamma\) FILTER**

For the Kalman filter in steady-state conditions, \(P_{k|k} = P_{k-1|k-1}\), and \(P_{k+1|k} = P_{k|k-1}\), and \(K_k = K_{k-1}\). For a Kalman filter to achieve these steady-state conditions, the error processes, \(w_k\) and \(n_k\), must have stationary statistics and the data rate must be constant. The \(\alpha, \beta, \Gamma\) filter is the steady-state Kalman filter for tracking nearly constant acceleration targets.

The \(\alpha, \beta, \Gamma\) filter with piecewise constant jerk maneuver model is a single coordinate filter that is based on the assumption that the target is moving with constant acceleration plus zero-mean, white Gaussian jerk errors which are constant from time \(k - 1\) to time \(k\). Given this assumption, the \(\alpha, \beta, \Gamma\) filter gains are chosen as the steady-state Kalman gains that minimize the mean-square error in the position, velocity, and acceleration estimates. For the \(\alpha, \beta, \Gamma\) filter, \(\alpha, \beta, \Gamma\) filter gains are chosen as the steady-state Kalman gains that minimize the mean-square error in the position, velocity, and acceleration estimates. For the \(\alpha, \beta, \Gamma\) filter,

\[
X_k = [x_k \ x_k \ x_k]^T \quad (8)
\]
\[
F_k = \begin{bmatrix} 1 & T & 0.5T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \quad (9)
\]
\[
G_k = \begin{bmatrix} T^3 \\ 6 \\ T^2 \end{bmatrix} \quad (10)
\]
\[
H_k = H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad (11)
\]
\[
R_k = R = \sigma_v^2 \quad (12)
\]
\[
Q_k = Q = \sigma_w^2 \quad (13)
\]

\[
K_k = K = \begin{bmatrix} \alpha & \beta & \Gamma \end{bmatrix}^T. \quad (14)
\]

Let the steady-state error covariance matrix of the filtered estimates for the \(\alpha, \beta, \Gamma\) filter be denoted as

\[
P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}. \quad (15)
\]

Using (5), (11), and (12) gives the steady-state gain as

\[
K = PH^T\sigma_v^{-2} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} \sigma_v^{-2}. \quad (16)
\]

Thus, using (14) gives

\[
p_{11} = \alpha \sigma_v^2 \quad (17)
\]
\[
p_{12} = \frac{\beta}{T} \sigma_v^2 \quad (18)
\]
\[
p_{13} = \frac{\Gamma}{T^2} \sigma_v^2. \quad (19)
\]

Inserting (4) into (7) and setting \(P_{k|k} = P_{k-1|k-1} = P\) for steady-state conditions gives

\[
[I - KH]^{-1}P = FPF^T + GG^T\sigma_w^2 \quad (20)
\]

where

\[
[I - KH]^{-1}P = \begin{bmatrix} \frac{\alpha}{1 - \alpha} & \frac{\beta}{T} & \frac{\Gamma}{T^2} \\ \frac{\beta}{T} & \frac{\beta^2}{T^2} + (1 - \alpha)\frac{P_{22}}{\sigma_v^2} & \frac{\beta \Gamma}{T^3} + (1 - \alpha)\frac{P_{23}}{\sigma_v^2} \\ \frac{\Gamma}{T^2} & \frac{\beta \Gamma}{T^3} + (1 - \alpha)\frac{P_{23}}{\sigma_v^2} & \frac{\Gamma^2}{T^4} + (1 - \alpha)\frac{P_{33}}{\sigma_v^2} \end{bmatrix} \quad (21)
\]

\[
F_kPF_k^T = \begin{bmatrix} p_{11} + 2Tp_{12} + T^2(p_{13} + p_{22}) + T^3p_{23} + \frac{T^4}{4}p_{33} & \cdots & \cdots \\ p_{12} + T(p_{13} + p_{22}) + \frac{3T^2}{2}p_{23} + \frac{T^3}{2}p_{33} & p_{22} + 2Tp_{23} + T^2p_{33} & \cdots \\ p_{13} + Tp_{23} + \frac{T^2}{2}p_{33} & p_{23} + Tp_{33} & p_{33} \end{bmatrix} \quad (22)
\]
Equating the (3,3) elements of (20) gives

$$\frac{\gamma^2}{1 - \alpha} = \frac{T^6 \sigma_u^2}{\sigma_v^2} = \Gamma^2$$  \hspace{1cm} (24)

where $\Gamma = T^3 \sigma_u/\sigma_v$ is the tracking index. Equating the (2,3), (2,2), and (2,1) elements of (20) and using (24) to eliminate $\sigma_u^2$ results in the following representation for the steady-state error covariance matrix. Equating the (3,1) and (1,1) elements of (20), along with (24), gives two additional equations used to find the $\alpha, \beta, \Gamma$ gains as

$$\beta^2 = \frac{\psi^2}{T^2} \left( 2 \alpha - \frac{\Gamma}{12} \right)$$  \hspace{1cm} (26)

$$0 = \beta \left( 2 - \frac{\beta}{4} - \alpha \right) - \alpha^2 + \frac{\Gamma^2}{144}$$  \hspace{1cm} (27)

The gains for the $\alpha, \beta, \Gamma$ filter are calculated by solving (24), (26), and (27) simultaneously. An expression in $\alpha$ with coefficients as a function of $T$ is given by

$$\alpha^4 + \alpha^3 \left( -\frac{5T^2}{18} \right) + \alpha^2 \left( \frac{16T^4}{144} + \frac{11T^2}{18} \right) + \alpha \left( -\frac{32T^4}{144} - \frac{2T^2}{3} \right) + \frac{16T^4}{144} + \frac{T^2}{3} + \Gamma (1 - \alpha)^{1/2} - (\alpha^3 + 8\alpha^2 - 8\alpha) - T^3 (1 - \alpha)^{3/2} \left( \frac{a}{36} \right) = 0.$$  \hspace{1cm} (28)

A zero of (28) is easily found by utilizing Newton's method when given the tracking index $T$ and a good initial guess of $\alpha$. The $\beta$ and $\Gamma$ gains are then calculated using (24) and (26). The MATLAB code for computing the gains when given a tracking index is presented in Table I, where the following third order polynomial,

$$\alpha_0 = -0.0001 \cdot \left( \log(T) \right)^3 + 0.0095 \cdot \left( \log(T) \right)^2 + 0.18 \cdot \log(T) + 0.71$$  \hspace{1cm} (29)

is utilized to provide an initial guess of $\alpha$, denoted as $\alpha_0$. The results of this process yield the same gains and error covariance as [1].

**SUMMARY**

Simple relationships for the gains $\alpha, \beta,$ and $\Gamma$ and the steady-state error covariance were presented for a system with piecewise constant jerk input noise. The relationships for the gains were used in a MATLAB program to calculate the steady-state gains for a given tracking index. Once the gains are obtained, the terms of the steady-state error covariance matrix are readily obtained. These relationships are much simpler than those presented in [1], while providing the same results.

**REFERENCES**


TABLE I
Gain Program

| % This file uses a Newtonian iteration |
| % technique to solve for the |
| % alpha, beta, gamma filter gains. |

function [alpha, beta, gamma] = gainabg(ti)

tf = log10(ti);

% initial guess for alpha
if tf > 1.5
    al = .999;
elseif tf < -9
    al = .0005;
else
    al = -1.0e-4*tf^3+9.5e-3*tf^2+.18*tf+.71;
end

% Newton's method
g = 0; delta = 10;
c3 = -(ti*ti^2/18);  
c2 = (-32*ti^4/144*2); 
c1 = (16*ti^4/144*2)+(ti^2/3);
c0 = (16*ti^4/144*2)+(ti^2/3);

while delta > .0001
    g = g + 1;
    rad = ti^2 - ti^2*al; 
    % calculate the function at al
    fa1 = al^4+c3*al^3+c2*al^2+c1*al+c0+ ... 
        (sqrt(rad)) + ... 
        (-al^3+(8*al^2)-(8*al)) + ... 
        (rad^-1.5)*(-al/36));
    % calculate the derivative at al
    faip = 4*al^3+3*c3*al^2+2*c2*al+c1+ ... 
        (sqrt(rad)) + ... 
        ((al^2/24)-(3*al)+(16*al)-8)) + ... 
        (rad^-1.5)*(-1/36));
    % calculate the new value of alpha
    a2 = a1 - (fa1 / faip);
    delta = abs(a2 - a1);
    if g > 20
        break;
    end;
    a1 = a2;
end;

alpha = sqrt(t1^2*(1-alpha));
gamma = sqrt((2*alpha*gamma)-(gamma^2/12));

A Variable Projection Method for Additive Components with Application to GPS

A variable projection method is presented for the case of additive linear components. Specifically, the least-squares problem of minimizing $||x - f(\phi) - L\lambda||^2$ over parameters $(\lambda, \phi)$ is reduced to that of minimizing $||F^T(x - f(\phi))||^2$ over $\phi$, where $F$ is an $(n-p) \times n$ projection orthogonal to $L$, $p$ being the dimension of $\lambda$. Applications to signal modeling and Global Positioning System (GPS) navigation are given. In the case of GPS, it is shown that least-squares point position estimates based on pseudoranges are equal to those based on pseudorange differences.

I. INTRODUCTION

Many signal modeling and parameter estimation problems can be formulated in terms of a least squares optimization. It is often the case that the error to be minimized has both linear and nonlinear components. The linear components may appear multiplicatively, as in the case of estimating the amplitudes and frequencies of a set of sinusoids in noise:

$$\min_{\lambda, \phi} ||x - F(\phi)\lambda||^2$$

(1)

where $x$ is the measured signal, $F$ represents the signal basis, parameterized by the unknown frequencies $\phi$, and $\lambda$ is the set of unknown amplitudes. The linear components may also appear additively, i.e.,

$$\min_{\lambda, \phi} ||x - f(\phi) - L\lambda||^2$$

(2)

as in the case of removing a trend before modeling.

In the multiplicative case (1), the complexity of the problem can be reduced by eliminating the linear parameters $\lambda$ using a technique due to Golub and Pereyra called variable projection [1]. Note that the quantity

$$\lambda^* = (F^T F)^{-1} F^T x$$

(3)

minimizes $||x - f(\phi)\lambda||^2$ for any choice of $\phi$. Substituting $\lambda^*$ into (1), the optimal nonlinear parameters $\phi^*$ are seen to be those minimizing the norm of the data $x$ projected onto the space orthogonal to $F$,

$$\phi^* = \text{Arg} \left[ \min_{\phi} \{||P_F^T x||^2\} \right]$$

(4)

$$P_F^T = I - F(F^T F)^{-1} F^T.$$

Manuscript received June 5, 1993.

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