with respect to $K$ one obtains the following constraints.

\[ T^6 \sigma^2 / \sigma^2_{\mu} = \Gamma^2 / (1 - \alpha) \]
\[ \beta = 3(2 - \alpha)(24\alpha - \Gamma) / (36\alpha^2 + 18\alpha\Gamma - \Gamma^2) \]
\[ 0 = 1296\alpha^4 - 10368\alpha\Gamma + 10368\alpha^2\gamma - 1296\alpha^3\Gamma + 432\Gamma^2 - 432\alpha\Gamma^2 + 360\alpha^2\Gamma^2 - 360\alpha^3 + \Gamma^4. \]

Solving the third equation we have

\[ \Gamma = \left( S - UW \right) \left( -4 - 432 + 864\alpha + 1440\alpha^2 - 1872\alpha^3 ight. \\
- 12\alpha^4 + 72U - 72\alpha U + 60\alpha^2 U - 12U^2 \\
- UW \right) / U + S^2 / \left( U^2 W^2 \right)^{1/2} / (2UW) \]

where the following definitions are used.

\[ R = \left[ \left[ 1 - \alpha \right] \left[ 144 - 288\alpha + 357\alpha^2 - 213\alpha^3 + 2\alpha^4 \right] \right]^{1/2} \]
\[ U = \left\{ 216 + 648\alpha - 3132\alpha^2 + 5184\alpha^3 - 1989\alpha^4 \\
- 495\alpha^5 + \alpha^6 - 27(4\alpha - 4\alpha^2 + \alpha^3)\right\}^{1/3} \]
\[ W = \left\{ -12\alpha^4 U^2 + \left[ (6 - 6\alpha + 5\alpha^2)U - (36 - 72\alpha \\
- 120\alpha^2 + 156\alpha^3 + \alpha^4) - U^2 \right]^2 \right\}^{1/3} / \alpha \]
\[ S = 18\alpha (432 - 864\alpha + 1440\alpha^2 - 1872\alpha^3 + 12\alpha^4 + 216U \\
- 216\alpha U - 24\alpha^2 U + 12U^2 + UW). \]

A tracking index can be defined (see [1]) as

\[ T^2 = T^6 \sigma^2 / \sigma^2_{\mu} = \Gamma^2 / (1 - \alpha). \]

**Step 6 Plot $\alpha$, $\beta$, $\Gamma$ as Function of Tracking Index:**

Fig. 1 shows the values of $\alpha$, $\beta$, and $\Gamma$ as a function of $\log[T^2]$. Note that in the limit of $T$ going to infinity, $\alpha$ has a limit of 1, $\beta$ has a limit of $\sqrt{3}$, and $\Gamma$ has a limit of $(12 - 6\sqrt{3})$.

**CONCLUSION**

For the set of physical problems where the model error dynamic can be approximated as a noisy jerk, a set of optimal relationships have been derived between the gains $\alpha$, $\beta$, $\Gamma$ and the tracking index of an $\alpha$-$\beta$-$\Gamma$ filter. The covariances are also computed. At the optimal relationships, some covariance elements can be expressed in a simple equation.

\[ P_{11} = \alpha \sigma^2_{\mu} \]
\[ P_{12} = \beta \sigma^2_{\mu}/T \]
\[ P_{13} = \Gamma \sigma^2_{\mu}/T^2. \]

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**General Expressions for Rician Density and Distribution Functions**

General expressions are derived for the density and distribution functions of the amplitude of a linear matched filter output given a nonfluctuating target in arbitrary clutter. These generalized Rician density and distribution functions are based on the density function of the clutter amplitude at the receiver output, and are easily evaluated numerically. Furthermore, we present error bounds for the numerical calculation of the density function. We also present several numerical examples.

**I. INTRODUCTION**

The reliable detection of a target in clutter is perhaps the most important objective in the development of a practical echo-location system. The classic theory of detection, that based on the detection of a fluctuating or nonfluctuating target imbedded in complex Gaussian noise, is quite mature. In particular,
a well-documented portion of this theory addresses
detection via multiple observations based on the
Swerling target models [1, 2]. However, since the
development of these models, the resolution of radar
and sonar systems has increased, thereby implying that
the physical size of an illumination cell has decreased.
The result has been the appearance of non-Gaussian
clutter [3]. Moreover, the reduction in illumination
cell size implies that one can resolve a single target,
or single highlight of a target. These highlights can
be considered nonfluctuating; the amplitude of a
return from the highlight remains constant. Under
these conditions, application of the classic theory can
prove disappointing. For example, in the case of binary
detection, the application of the classic theory yields
expressions to determine a threshold value for a given
false alarm rate. In practice, however, the use of this
threshold in the presence of non-Gaussian clutter can
cause a detection system to display a false alarm rate
that is higher than that predicted by the classic theory.

Almost all detection analysis requires the density
function and/or distribution function for the value
of the receiver output. Commonly, the receiver
is a matched filter whose structure is shown in
Fig. 1. The density and distribution functions that
describe the amplitude of a matched filter detecting
a nonfluctuating target in Gaussian clutter are called
Rician, or more specifically Rayleigh-Rician. Here
we present integral expressions to determine these
functions given an arbitrary clutter probability density
function. It should be noted that work similar to this
has been done previously [4-6]; however, there is some
novelty to the results we present here. First, previous
integral expressions for the distribution function
require the distribution of the clutter. Although in
a theoretical context this should not matter, there
exist density functions, such as the lognormal density,
that do not have distribution functions expressible
in closed form. The integral expression for the
distribution function presented here requires only
the density function in its integrand, thus precluding
a more intense numerical approach requiring a double
integration. Second, the expressions presented here
are slightly more streamlined; they do not invoke a
nonlinear transformation of the argument of the clutter
density function as do previous expressions. Finally, we
present error bounds for the numerical evaluation of
the integral expression of the distribution function.

II. GENERAL RICIAN DENSITY FUNCTION

If \( f(t) \) is the transmitted signal whose energy is \( E \),
then we model the return waveform as
\[
x(t) = r(t)e^{j\phi}f(t - \tau)e^{j\omega t} + n(t)
\]
where \( n(t) \) is the clutter return, and \( r(\tau) \) and \( \phi(\tau) \)
are, respectively the amplitude and phase of the

reflection from the target whose range delay is \( \tau \) and
motion-induced Doppler shift in radian frequency is \( \omega \). As shown in Fig. 1, the return is processed
by a matched filter receiver which calculates the
integral correlation between the return and an
energy-normalized ideal version of the transmit signal
subjected to a hypothesized range delay \( \hat{\tau} \) and Doppler
shift \( \hat{\omega} \). The output of the correlator is followed by an
envelope detector to yield the receiver output \( |l(\hat{\tau},\hat{\omega})| \).
In practice, one calculates the value of \( |l(\hat{\tau},\hat{\omega})| \) over
a range of hypothesized range delays and Doppler
shifts at discrete points or ‘bins’ in range/Doppler
space. However, our analysis assumes that we are
addressing the bin occupied by the target. In this case,
the matched filter output prior to envelope detection is
the complex value
\[
l = r e^{j\phi} = r_T e^{j\phi_T} + r_C e^{j\phi_C}
\]
where \( r_C \) and \( \phi_C \) are, respectively, the amplitude and
phase of the matched filter response to the clutter.
We make the common assumption that we have no
knowledge of the target and clutter phase. Hence, both
\( \phi_T \) and \( \phi_C \) are taken to be uniform random variables
in the interval \((0, 2\pi)\). This implies that for both target
and clutter, the joint density function of amplitude and
phase is circularly symmetric. Therefore, using \((r,\phi)\) as
a general polar notation, the density functions for both
target and clutter are, respectively,
\[
\begin{align*}
f_{R,\phi}(r,\phi) &= \frac{1}{2\pi} f_n(r) \delta(\phi - \phi_T) \\
f_{R,\phi}(r,\phi) &= \frac{1}{2\pi} f_n(r) \delta(\phi - \phi_C)
\end{align*}
\]

The right side of (2) shows that the matched filter
output is a sum of two complex random variables.
As in the case of the sum of two real-valued random
variables, the probability density function for the
complex value of the matched filter output \( r e^{j\phi} \) is the
convolution of the probability density functions for
\( r_T e^{j\phi_T} \) and \( r_C e^{j\phi_C} \). This convolution is two-dimensional,
and is performed in the equivalent Cartesian
coordinates associated with the real and imaginary
parts of \( r e^{j\phi} \). To use this approach directly implies that
we must invoke a transformation of random variables
from polar to rectangular coordinates. However, this
can be circumvented by using a different method
developed by Jakeman and Pusey [7], and Jao [8, 9].
In this approach we apply the Fourier transform to
find the characteristic functions of \( f_{R,\phi}(r,\phi) \) and
\( f_{R,\phi}(r,\phi) \), find their product, and invoke the inverse
transform to yield the density function for the matched
filter output, \( f_{R,b}(r, \theta)_{T+C} \). This approach is further simplified by exploiting the circular symmetry of the density functions. In this case, it can be shown that the characteristic functions for \( f_{R,b}(r, \theta)_{T} \) and \( f_{R,b}(r, \theta)_{C} \) are also circularly symmetric, and, by application of the forward Hankel transform [10], are given by

\[
\Phi(\rho)_{T} = \int_{0}^{\infty} f_{R}(r)_{T} J_{0}(\rho r) \, dr \tag{4}
\]

\[
\Phi(\rho)_{C} = \int_{0}^{\infty} f_{R}(r)_{C} J_{0}(\rho r) \, dr \tag{5}
\]

where \( J_{0}(\cdot) \) is the Bessel function of order zero. The characteristic function for \( f_{R,b}(r, \theta)_{T+C} \) is then

\[
\Phi(\rho)_{T+C} = \Phi(\rho)_{T} \Phi(\rho)_{C}. \tag{6}
\]

This function is circularly symmetric implying that \( f_{R,b}(r, \theta)_{T+C} \) is also circularly symmetric. Therefore, application of the inverse Hankel transform of (6) yields the density function of the magnitude of the matched filter output, i.e.,

\[
f_{R}(r)_{T+C} = r \int_{0}^{\infty} \rho \Phi(\rho)_{T+C} J_{0}(\rho r) \, d\rho. \tag{7}
\]

Since we have assumed that the target is nonfluctuating, the density function for the magnitude of the target return is

\[
f_{R}(r)_{T} = \delta(r - r_{0}) \tag{8}
\]

where \( \delta(\cdot) \) is the Dirac delta function. Substituting (8) into (4) yields

\[
\Phi(\rho)_{T} = J_{0}(\rho r_{0}). \tag{9}
\]

Combining (4) to (7) and (9) yields

\[
f_{R}(r)_{T+C} = r \int_{0}^{\infty} f_{R}(a)_{C} \times \left[ \int_{0}^{\infty} \rho J_{0}(\rho a) J_{0}(\rho r_{0}) J_{0}(\rho r) \, d\rho \right] da. \tag{10}
\]

The bracketed term in the integrand of (10) can be found in closed form [11, 12]:

\[
\int_{0}^{\infty} \rho J_{0}(\rho a) J_{0}(\rho r_{0}) J_{0}(\rho r) \, d\rho = \frac{1}{2\pi \Delta(a, r_{0}, r)} \tag{11}
\]

where \( \Delta(a, r_{0}, r) \) is the area of a triangle whose sides are of lengths \( a, r_{0}, \) and \( r. \) If a triangle cannot be formed, say when \( a > r_{0} + r, \) then the integral in (11) is zero. Furthermore, an explicit form for \( \Delta(a, r_{0}, r) \) can be found using Heron's formula [13] given by

\[
\Delta(a, r_{0}, r) = \frac{1}{4} \sqrt{(a + r_{0} + r)(a + r_{0} - r)(a - r_{0} + r)(-a + r_{0} + r)} \tag{12}
\]

which applies when \( |r - r_{0}| < a < r_{0} + r. \) Substitution of (11) and (12) into (10) yields the final form of the general Rician density function as

\[
f_{R}(r)_{T+C} = \frac{2r}{\pi} \int_{|r - r_{0}|}^{\infty} \frac{f_{R}(a)_{C} \, da}{\sqrt{(a + r_{0} + r)(a + r_{0} - r)(a - r_{0} + r)(-a + r_{0} + r)}} \tag{13}
\]

\[\text{III. ERROR BOUNDS}\]

In all likelihood the integral in (13) will not admit a closed-form solution and will have to be evaluated numerically. This, in turn, presents another problem since the integrand of (13) has poles at the limits of integration \( |r_{0} - r| \) and \( r_{0} + r. \) To handle this, we adjust the limits so that (13) is well approximated by

\[
f_{R}(r)_{T+C} \approx \frac{2r}{\pi} \int_{|r - r_{0}| + \epsilon}^{\infty} \frac{f_{R}(a)_{C} \, da}{\sqrt{(a + r_{0} + r)(a + r_{0} - r)(a - r_{0} + r)(-a + r_{0} + r)}} \tag{14}
\]

for some \( \epsilon > 0. \) Naturally we choose \( \epsilon \) small enough so that \( |r_{0} - r| + \epsilon < r_{0} + r - \epsilon. \) Since we are now approximating \( f_{R}(r)_{T+C} \) by eliminating a portion of the integral in (13) at each of its limits, it is of interest to determine how much error this will introduce. There are four cases of error we can examine. Bounds for these errors are presented below.

**Lower Limit Error Bound, \( r \neq r_{0} \):** Consider eliminating the portion of the integral in (13) at the pole at the lower limit of integration for \( r \neq r_{0}. \) The error for this truncation of the integral is

\[
E_{-} = \frac{2r}{\pi} \int_{|r - r_{0}|}^{\infty} \frac{f_{R}(a)_{C} \, da}{\sqrt{(a + r_{0} + r)(a + r_{0} - r)(a - r_{0} + r)(-a + r_{0} + r)}} \tag{15}
\]

A bound for \( E_{-} \) can be found by realizing that both the numerator and denominator of the integrand are nonnegative over the region of integration, thus

\[
E_{-} \leq \frac{2r}{\pi} \max_{y \in [0, 1]} \frac{f_{R}(|r_{0} - r| + y)c}{\sqrt{(r_{0} + r + |r_{0} - r|)(r_{0} + r - |r_{0} - r| - \epsilon)}} \times \int_{|r_{0} - r|}^{\infty} \frac{da}{\sqrt{a^{2} - |r_{0} - r|^{2}}} \tag{16}
\]

\[\text{where} \quad \epsilon = B_{-} \].
Note that we must have \( r_0 + r - |r_0 - r| - \epsilon > 0 \). This is always true for \( r_0, r > \epsilon/2 \). If this were not true, then the bound would be meaningless since it would be an infinite or a complex value.

Upper Limit Error Bound, \( r \neq r_0 \): Consider eliminating the portion of the integral in (13) at the pole at the upper limit of integration for \( r \neq r_0 \). The error for this truncation of the integral is

\[
E_+ = \frac{2r}{\pi} \int_{r_0 + \epsilon}^{r_0 + \epsilon} \frac{g(r)c}{\sqrt{(a + r_0 + r)(a + r_0 - r)(a + r_0 + r)(-a + r_0 + r)}} da
\]

A bound for \( E_+ \) can be found by using the same approach used to find the bound for \( E_- \), thus

\[
E_+ \leq \frac{2r}{\pi} \max_{y \in [0,1]} g(r_0 + r - yc) \cdot \frac{f_r(2r_0 + 3r_0 - 3r_0 - \epsilon)}{(2r_0 - \epsilon)(2r_0 + \epsilon)} \int_{r_0 + \epsilon}^{r_0 + \epsilon} \frac{da}{\sqrt{r_0 + \epsilon - a}}
\]

\[
= \frac{4r}{\pi} \max_{y \in [0,1]} g(r_0 + r - yc) \cdot \frac{f_r(2r_0 + 3r_0 - 3r_0 - \epsilon)}{(2r_0 - \epsilon)(2r_0 + \epsilon)} \sqrt{\epsilon}
\]

\[
= B_+ \quad \text{(18)}
\]

Note that we must have \( r_0, r > \epsilon/2 \) for the bound to be a real finite value.

Lower Limit Error Bound, \( r = r_0 \): Consider eliminating the portion of the integral in (13) at the pole at the lower limit of integration for \( r = r_0 \). The error for this truncation of the integral is

\[
E_- = \frac{2r_0}{\pi} \int_{r_0}^{r_0} \frac{g(r)c}{\sqrt{4r_0^2 - a^2}} da.
\]

Unlike the two previous cases, we do not present a closed-form expression for a bound since the integrand denominator introduces a nonintegrable pole at zero. Therefore, let \( g(r) \) and \( h(r) \) be integrable functions such that

\[
g(r) \leq \frac{f_r(r)c}{a} \leq h(r)
\]

for \( 0 \leq r \leq \epsilon \). Furthermore, note that

\[
1 \leq \frac{2r_0}{\sqrt{4r_0^2 - a^2}} \leq \frac{2r_0}{\sqrt{4r_0^2 - \epsilon^2}}
\]

for \( 0 \leq a \leq \epsilon \). By combining (19) to (21) \( E_-^0 \) can be bounded from above and below by the inequality

\[
B_{1-} = \frac{1}{\pi} \int_{0}^{r_0} \frac{g(a)c}{a} da \leq E_-^0 \leq \frac{2r_0}{\pi \sqrt{4r_0^2 - \epsilon^2}} \int_{0}^{r_0} \frac{h(a)c}{a} da = B_{1-}^0
\]

For this inequality to be meaningful we must have \( r_0 > \epsilon/2 \). This ensures that \( B_{1-}^0 \) is a nonnegative real value.

Upper Limit Error Bound, \( r = r_0 \): Consider eliminating the portion of the integral in (13) at the pole at the upper limit of integration for \( r = r_0 \). The error for this truncation of the integral is

\[
E_0 = \frac{2r_0}{\pi} \int_{r_0 - \epsilon}^{r_0} \frac{f_r(r)c}{\sqrt{4r_0^2 - a^2}} \cdot \frac{da}{a}.
\]

A bound for \( E_0^0 \) is

\[
E_0^0 \leq \frac{2r_0}{\pi} \max_{y \in [0,1]} f_r(2r_0 - y)c \int_{r_0 - \epsilon}^{r_0} \frac{da}{\sqrt{4r_0^2 - a^2}}
\]

\[
= \frac{1}{\pi} \max_{y \in [0,1]} f_r(2r_0 - y)c \ln \left( \frac{2r_0 + \sqrt{4r_0^2 - \epsilon^2}}{2r_0 - \epsilon} \right)
\]

\[
= B_0^0.
\]

For the argument of the natural logarithm to be a real value greater than 1, we must have \( r_0 > \epsilon/2 \). This implies the bound will also be a nonnegative real value.

IV. THE GENERAL RICIAN DISTRIBUTION FUNCTION

In detection performance calculations, one does not usually work directly with the density function, but uses the distribution function defined as

\[
F_r(r)_{\tau+c} = \int_{0}^{r} f_r(t)_{\tau+c} dt
\]

and is zero for \( r < 0 \). With this function one can find quantities such as the probability of false alarm (\( P_a \)), and the probability of target detection (\( P_d \)). To find the distribution function we could simply numerically integrate (14); however, this would involve a double integration. To avoid this computationally expensive approach, we seek an expression for the distribution function that would involve only one integration. In this case, numerical computation of the distribution function would, in principle, be no more difficult than the numerical computation of the density function in (14).

We begin by substituting (10) into (25), and interchanging the order of integration to get

\[
F_r(r)_{\tau+c} = \int_{0}^{r} f_r(r)_{\tau+c} \int_{0}^{r} \rho J_0(\rho \gamma) J_0(r_0 \gamma) d\rho \frac{dz}{z}.
\]

However, it can be shown that the innermost bracketed term in (26) is given by

\[
\int_{0}^{r} z J_0(\rho \gamma) d\gamma = \frac{z}{\rho} J_1(\rho \gamma)
\]
where \( J_1(\cdot) \) is the Bessel function of order 1. Thus, substituting (27) into (26) yields

\[
F_R(r)\tau + c = r \int_0^\infty f_R(a)c \\
\times \left[ \int_0^\infty J_0(pa)J_0(p\rho)J_1(pr)\,dp \right] \,da.
\]

(28)

It can also be shown that the bracketed term in (28) has a closed form [11], and is given by

\[
\int_0^\infty J_0(pa)J_0(p\rho)J_1(pr)\,dp = \begin{cases} 
\frac{1}{\pi} \arccos \left( \frac{r^2 + a^2 - r^2}{2\rho a} \right) & r < |r_0 - a| \\
\frac{1}{r} & |r_0 - a| \leq r \leq r_0 + a \\
\frac{1}{\pi} \arccos \left( \frac{r_0^2 + a^2 - r^2}{2\rho a} \right) & r_0 + a < r
\end{cases}
\]

(29)

Substituting (29) into (28) yields

\[
F_R(r)\tau + c = F_R(r - r_0)c + \frac{1}{\pi} \int_{r_0 - r}^{r_0 + r} f_R(a)c \arccos \left( \frac{r_0^2 + a^2 - r^2}{2\rho a} \right) \,da \\
\times \int_{r_0 - r}^{r_0 + r} f_R(a)c \,da + \frac{1}{\pi} \\
\times \int_{r_0 - r}^{r_0 + r} f_R(a)c \arccos \left( \frac{r_0^2 + a^2 - r^2}{2\rho a} \right) \,da.
\]

(30)

Note that \( F_R(r - r_0) \) is zero for \( r - r_0 < 0 \). For the special case of \( r = r_0 \) the integrand in (30) can be simplified yielding

\[
F_R(r_0)\tau + c = F_R(0)\,c + \frac{1}{\pi} \int_0^{2\pi} f_R(a)c \arccos \left( \frac{a}{2\rho_0} \right) \,da.
\]

(31)

Note that both (30) and (31) use the probability density function in their integrand unlike an earlier approach that integrates the distribution function [5, 6]. Furthermore, \( F_R(0)\) appears in (31) as a mathematical formality to account for the existence of a mass point in \( f_R(r)c \) at \( r = 0 \).

As in the case for the expression for the density function, it is also likely that the integrals in (30) and (31) will not admit a closed-form solution and will have to be evaluated numerically. Because \(-1 \leq \frac{r_0^2 + a^2 - r^2}{2\rho a} \leq 1\), and \(0 \leq \arccos(y) \leq \pi\) for \(-1 \leq y \leq 1\), we see that we need not integrate over any singularities unless they are due to the density function. Therefore, we do not derive error bounds such as those presented in Section III.

V. NUMERICAL EXAMPLES

In this section we present examples of numerically integrating (13), (30), and (31). They demonstrate that in a practical problem the bounds derived in Section III can be "tight," and that accurate results can be obtained by using common numerical integration techniques. All examples were performed on a NeXT computer running Mathematica, a software package designed to do symbolic mathematics as well as numerical computations [14]. Where appropriate, we list in parentheses the Mathematica operations and their associated parameter values used in the numerical integration.

Mathematica uses an adaptive technique for numerically integrating functions (NIntegrate). It first samples the integrand at a sequence of points (Points->Automatic). If it finds that the integrand changes rapidly in a given region of its domain, then it recursively takes more samples in that region. There is a minimum and maximum number of recursions it will perform for resampling the integrand, and for these examples we used the default values of 1 and 6, respectively (MinRecursion->1, MaxRecursion->6). In no case did Mathematica indicate that it required more than six sampling recursions to achieve sufficient numerical accuracy. Although Mathematica is capable of performing calculations to an almost arbitrary number of decimal places, the numerical accuracy for all internal computations was set to the machine precision of the NeXT, which is 16 decimal places (WorkingPrecision->16). This is equivalent to double precision on most mainframe computers. The precision of the numerically calculated value of the integral in (13) was less than that of the internal computations; however, Mathematica attempted to achieve six digit accuracy (AccuracyGoal->6).

There are some cases where Mathematica is capable of integrating over a singularity, and in particular, it will always test for the presence of singularities at the upper and lower limits of integration. However, this feature was not exploited so that we could assess the quality of the error bounds derived in Section III, and obtain results comparable to those obtainable by less sophisticated numerical integration software.

In all the examples we assumed the clutter amplitude distribution for the matched filter output followed a Weibull distribution, i.e.,

\[
f_R(r)c = \frac{c}{b} r^{c-1} e^{-r^c/b^c}
\]

(32)

where \( b \) is the scale parameter, and \( c > 0 \) is the shape parameter that relates to the skewness of the distribution. Without loss of generality, the shape
parameter was chosen to be
\[ b = (\ln(2))^{-1/c} \]  
(33)
giving the distribution in (32) a median value of 1. The Weibull distribution is used extensively in reliability and failure rate analysis; however, it has found favor in detection theory since it provides a good model for the distribution of the matched filter amplitude response to non-Gaussian clutter [2, 4, 15, 16]. As shown in Fig. 2, for \( c < 2 \) the distribution contains more area under its 'tail' than does the Rayleigh distribution which models the matched filter amplitude response to Gaussian clutter. For example, it has been found experimentally that \( K_u \)-band radar irradiating the ocean surface at sea state 3 at a grazing angle of \( 1^\circ \) to \( 30^\circ \) will exhibit a matched filter response whose amplitude follows a Weibull distribution with \( 1.160 < c < 1.783 \) [4].

Three of the error bounds, \( B_0 \), \( B_\ast \), and \( B^\ast \), are already given in closed form. However, the error bounds \( B^0_L \) and \( B^0_U \), are related to the distribution \( f_R(r)C' \). Assuming \( f_R(r)C' \) is a Weibull distribution, these bounds are given as follows. For \( c > 1, r \) near zero, and \( \epsilon \) well below the median value of the Weibull distribution
\[ g(r) = \frac{C}{b^e} e^{-\epsilon/(b)^e} \leq f_R(r)C \leq \frac{C}{b^e} e^{-1} = h(r). \]  
(34)
Substituting (34) into (22) yields
\[ B^0_L = \frac{C e^{\epsilon-1} e^{-(c/b)^y}}{(c-1) e^{b^e}} \leq E^0 \]
\[ \leq -\frac{2r_0}{\sqrt{4r_0^2 - \epsilon (c-1) e^{b^e}}} = B^0_U. \]  
(35)
Note that these two bounds are very close for \( c = 0.00001 \); therefore,
\[ E^0 \approx B^0_L = \frac{C e^{\epsilon-1} e^{-(c/b)^y}}{(c-1) e^{b^e}} \]  
(36)
and we may add this to the value of the numerical integration of (14). As the examples show, \( E^0 \) can be a significant fraction of the integral in (14). For \( c \leq 1 \), we may choose
\[ g(r) = \frac{C e^{-(c/b)^y}}{\pi b^e} \]  
(37)
which implies
\[ f_R(r)C = \frac{C e^{-(c/b)^y} \int_0^r da}{\pi b^e} = \infty \]  
(38)
and proves that \( f_R(r)C = \infty \) for \( c \leq 1 \).

For our first example we used \( c = 0.8 \), and the value of the nonfluctuating target amplitude \( r_0 \) was equal to 1, 2, 4, and 8. Furthermore, we set \( \epsilon = 0.00001 \). The results are shown in Fig. 3, and are composed of the significant values of the curves for \( 0 \leq r \leq 12 \). Also shown in the figure is the curve for \( r_0 = 0 \). This corresponds to only clutter being present, and is the curve \( f_R(r)C \). Furthermore, due to the high slope of the curve near \( r = r_0 \), each curve was evaluated at points given by
\[ r_m = m\Delta r - \frac{1}{1 + e^{-3.85(m\Delta r - r_0)}} + \frac{1}{2} \]  
(39)
where \( m \) is an integer and \( \Delta r = 0.2 \). This caused the numerically calculated curve of \( f_R(r)C \) to be "sampled" more frequently near the peak of the distribution at \( r = r_0 \) resulting in a smoother plot. In fact, in all cases the peak value is infinite at \( r = r_0 \) since \( c \leq 1 \). However, Mathematica recognized the overflow when it calculated \( f_R(r_0)C \) and indicated that an infinite value had occurred (Infinity). If less sophisticated software is used, then one can simply avoid numerically integrating for \( r = r_0 \).

For the second example we used \( c = 1.2 \). Again, the value of the nonfluctuating target amplitude \( r_0 \) was equal to 0, 1, 2, 4, and 8, and \( \epsilon = 0.00001 \). The results are shown in Fig. 4, and arc composed of the significant values of the curves for \( 0 \leq r \leq 12 \). As in the first example, the curves contain peaks at
1.4
1.2
0.8
0.6
Fig. 4. Family of Weibull-Rician density functions for $c = 1.2$, $r_0$ equal to 0, 1, 2, 4, 8.

Accordingly, these peaks are finite since $c > 1$.

For both examples, the lower and upper limit error bounds for $r \neq r_0$, $B_- \leq B_+$, were within 1% of the calculated values of the density functions. Furthermore, the upper limit error bound for $r = r_0$, $B_+^0$, was well below 0.1% of the calculated value of the density functions. Regarding the lower limit error for $c = 1.2$ and $r = r_0$, the absolute difference of its upper and lower bounds, $|B_+^0 - B_-^0|$, was on the order of $10^{-7}$. Furthermore, $B_-^0$ was approximately equal to 10% of the calculated value of the density function. This justified the approximation in (36), and adding its value to the numerical integration of (14) to yield $f_R(r_0)$.

To assess how accurately one can calculate a Rician density by numerically integrating (14), our last example replicates a result that is known in closed form. Here, we used

$$ f_R(r)^c = \frac{2 \pi^2}{b^2} r e^{(r/b)^2} $$

which is the Rayleigh distribution, and is a special case of the Weibull distribution when $c = 2$. This corresponds to clutter being modeled as a complex Gaussian noise process. In this case, it is well known that

$$ f_R(r) + c = \frac{2r}{b^2} e^{-(r/b)^2} \int_0^{2r/b} e^{u^2} du $$

and is referred to as the Rayleigh-Rician distribution [1, 2]. Again, the value of the nonfluctuating target amplitude $r_0$ was equal to 0, 1, 2, 4, and 8, and $\epsilon = 0.00001$. The results are shown in Fig. 5, and are composed of the significant values of the curves for $0 \leq r \leq 12$. Because these curves have no sharp peaks, they were evaluated at $r_m = 0.2m$ where $m$ is an integer. Over all values of $r_m$, the numerically calculated probability density function were within 1% of the value given by (41).

The distribution functions for the families of density functions presented in the three previous examples were also calculated using (30) and (31).

For $r_0 = 0$, corresponding to only clutter present, the distribution has the closed form

$$ F_R(r)^c = 1 - e^{-(r/b)^2} $$

The results are plotted in Figs. 6-8. In all cases the distribution functions were calculated at the same values of $r = r_m$ used to calculate the density functions. For the case of $c = 0.8$ (Figs. 2 and 5) the density function $f_R(r)^c$ approaches infinity as $r \to 0^+$ implying the integrands of (30) and (31) go to infinity at the lower limit of integration. However, *Mathematica* was able to handle this by a change of variable of integration near $r = 0$, and provide a numerical value for the integral. To assess the accuracy of what can be obtained by numerical integration of (30) and (31), we found the distribution for the case of a Rayleigh clutter density by direct numerical integration of the Rayleigh-Rician density in (41). It was found that this value agreed to 5 decimal places with the value obtained by numerical integration of (30) and (31) for each value of $r$. 
VI. DISCUSSION

We have presented integral expressions for deriving general Rician density and distribution functions that describe the linear matched filter output when detecting a nonfluctuating target return imbedded in arbitrary clutter. Numerical examples were performed to demonstrate the utility of the expressions, and assess the accuracy one can obtain even when numerical integration must be performed, which in practice would almost always be necessary.

The density and distribution functions obtained by the methods presented here can be applied to several other problems. First, and perhaps the most obvious application of the distribution function is the accurate calculation of the probability of missing a target ($P_m$) and the probability of target detection in a binary detection receiver. Second, there recently has been considerable research into the development of constant false alarm rate (CFAR) detection methods that are based on order statistics (OS-CFAR) [17]. Since the density function of an OS requires knowledge of both the density and distribution function of the matched filter output in each bin, application of the methods presented in this paper can be used to perform an accurate analysis of the performance of OS-CFAR given various clutter distributions. Third, one often does not base a detection on a single observation, but on multiple detections. In this case the test statistic is equal to the sum of the observations, and the density function for the test statistic is equal to the convolution of the density functions for each observation. If $n$ observations are assumed to be independent and identically distributed, then the density function for the test statistic is equal to an $n$-fold convolution of the distribution of a single observation. This can be accomplished via a fast Fourier transform (FFT) approach and the discrete uniform sampling of the Rician density functions derived from the expression presented here. Finally, suppose that the density function of the clutter is not known, but that a normalized histogram can be found based on experimental data. An approximate form of $f_R(r)c$ can be found by fitting a curve to this histogram. In turn, approximate density and distribution functions for the matched filter output could be obtained by using this measured form of $f_R(r)c$ and the expressions presented here.

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Azimuthal Scattering Pattern of Trees at X-Band

The azimuthal scattering characteristics of a variety of fully foliated trees were measured in the field using a short-range X band bistatic radar system operating at 9 GHz. Azimuthal measurements were made for both VV and VH polarizations over a 180° angle between the transmitter and receiver antennas at 10° intervals. Data were collected at transmit and receive ranges of approximately 5 m, and the beam impinged horizontally on the trees. The azimuthal power pattern, normalized to the copolarized backscattered power, showed a peak in the forward scattering direction (azimuth angle of 180°) due to direct coherent transmission, whose magnitude was seen to depend on the foliage area density. The diffuse bistatically scattered power in the range of azimuth angles between 10° and 90° showed standard deviations of the order of 4 dB. In addition, needleleaf trees were found to depolarize the incident energy more than the broadleaf trees, with depolarization ratios of the order of −2.2 dB and −6.3 dB, respectively, for bistatic scatter, and of the order of −4.5 dB and −8.9 dB, respectively, for backscatter.

I. INTRODUCTION

Although extensive data have been collected on the monostatic radar cross section (RCS) of various natural surface targets [1], bistatic data are extremely limited. The available database for ground targets consists of those obtained from just a few measurement programs [2–6]. The basic theory of bistatic radar was developed over three decades ago [7]; however, it is only recently that its full potential for various applications is being realized [8, 9].

The bistatic RCS of a surface clutter target is a measure of the energy scattered from the clutter cell area in the direction of the receiver. Bistatic RCS is thus a function of 1) the angle of incidence, θi, 2) the angle of scattering, θs, and 3) the azimuth angle between the transmitter and the receiver, φ. To characterize the bistatic scattering pattern of a target, therefore, comprehensive measurements are required for values of θi and θs between 0° and 90°, and φ between 0° and 360° (0° to 180° if the target is azimuthally symmetric). The monostatic scattering case is a special case of the bistatic configuration when φ = 0° and θs = θi, while the φ = 180° case refers to direct transmission through the medium. Since a large part of the Earth's surface is covered by trees, characterizing their bistatic scattering pattern is important in various radar and remote sensing applications. The results of our field measurements of the azimuthal scattering pattern of a variety of deciduous and coniferous trees are summarized here.

II. EXPERIMENTAL CONSIDERATIONS

A short-range CW bistatic radar system was used for these measurements. The system consisted of a HP 8720A Network Analyzer operating at 9 GHz, whose power output was amplified to 1 W in a Varian VTX 6180S1 travelling wave tube (TWT) amplifier. The receiver front end consisted of a 3.5 dB noise figure Miteq AMF-3B-8012-30 low-noise amplifier of 30 dB gain. Standard gain horn antennas of approximately 18° beamwidth were used for transmit and receive functions. The transmitter consisting of the network analyzer source, TWT amplifier, and transmit antenna was kept stationary, while the receive antenna and low-noise amplifier were mounted on a tripod and moved to various azimuthal positions. Both transmitter and receiver antennas were located at a height of approximately 1.5 m above ground level. A 20 m coxial cable was used between the output of the low-noise amplifier and the receiver section.