Low-Frequency Acoustic Wave-Scattering Phenomena Under Ice Cover

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Abstract—Studies on low-frequency acoustic wave-scattering phenomena due to under-ice roughness are made by utilizing a rough, thin-ice plate model. The scattering theory is based on Marsh’s [3] and Kuo’s [6] perturbation method. The under-ice roughness spectrum is based on Mellen’s empirical spectrum [10]. The model naturally divides the reflected field solution into specular and off-specular components.

The model for specular components can give an excellent propagation loss prediction if the combined effects of under-ice roughness scattering, ice absorption, and ice thickness are taken into account. The model for scattered or off-specular components is evaluated for a point source and point receiver geometry to study various spreading phenomena. It is found that time/angular/frequency spreads under ice cover are qualitatively similar to those caused by other types of rough boundaries. The Doppler frequency-shift spectrum is found to spread with the time of arrivals.

NOMENCLATURE

\( dA_{mnpq}, dB_{mnpq}, \) order differential amplitudes.
\( C_p, C_w \) acoustic velocities in air, water.
\( C_p \) complex longitudinal plate wave velocity.
\( F_{AA}, F_{BB}, F_{AB} \) scattering factors that are independent of roughness.
\( f_0, f \) source, received frequency.
\( g \) gravity acceleration.
\( H \) mean ice plate thickness.
\( H_0 \) source-receiver horizontal separation.
\( h_0 \) \( (H/2 + \eta)^3/2 \).
\( h_w \) \( (-H/2 + \eta)^3/2 \).
\( k_x, k_y \) \( \sqrt{\omega/C_p} \).
\( k_w \) \( \sqrt{\omega/C_w} \).
\( K_x, K_y \) \( (K_x, K_y) \).
\( K_a \) \( K_a(\lambda, \mu) \).
\( K_r \) \( K_r(\lambda, \mu) = K_s(\lambda, \mu) \).
\( K_w \) \( K_w(\alpha, \beta) \).
\( \hat{n}_a, \hat{n}_w \) unit normals at over-ice and under-ice roughness.
\( \rho_1, \rho_r \) incident, reflected pressures.
\( r_{10}, r_{20} \) specular path.
\( r_{12} \) off-specular path.

I. INTRODUCTION

This paper combines two reports by Kuo [1, 2]. Low-frequency acoustic wave scattering due to a rough surface has been analyzed by various perturbation methods originally proposed by [3]-[5]. The perturbation method utilized in this study is a generalized form of Marsh’s [3] and was originally proposed by Kuo [6].

Previous studies on propagation loss due to acoustic wave scattering for under-ice cover were based on an assumption of a pressure release boundary (e.g., [7] and [8]). Although theoretical predictions compared reasonably well with experi-

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mental data, theory slightly under-predicted the propagation loss. For long-range low-frequency propagation, the acoustic wave typically bounces and scatters at the under-ice cover quite frequently before it arrives at the receiver. Therefore a more accurate propagation loss estimate was essential. This is one of the reasons which prompted the study presented below.

The main objective of this study is to generate a simple model that is physically consistent. In this respect, the previous pressure-release models [7], [8] ignored the physical characteristics of the ice cover. The new theory based on a rough thin ice plate is not only an improvement in incorporating the physical characteristics of the ice cover, but also is physically appropriate for low-frequency acoustic waves. This new theory is formulated and solved in the next section.

Based on this solution, estimates of scattering loss and various spreads are made for various physical parametric values of the ice. The relative importance of these parameters is also assessed and discussed in the sections to follow.

Besides being an important acoustic wave for long-range propagation, a low-frequency acoustic wave permits the study of scattering phenomena by utilizing a wave perturbation theory, especially for small grazing angles. This is in contrast to a complex layered propagation model proposed by [15]. Because their model contains numerous physical parameters, it is quite difficult to uniquely associate any of those parameters with the observed propagation loss phenomena.

II. PROPOSED MODEL AND SOLUTION

Fig. 1 depicts the thin rough ice plate to be studied. In principle, both symmetrical and antisymmetrical modes of vibrations can be generated in the ice plate by an incident underwater acoustic wave. The antisymmetrical vibration (flexural wave) is generated by the acoustic pressure difference between the two sides of the plate, causing antisymmetrical loading. The symmetrical (longitudinal) wave is generated by the symmetrical part of the pressure loading. Because very little acoustic energy is transmitted into the air cover, the major pressure loading is antisymmetrical and the plate vibration is mainly flexural. In the following analysis, only antisymmetrical plate vibration is considered. It is also assumed that the flexural wavelength is large compared to the plate thickness; thus corrections for rotary inertia as well as shear deformation are not made.

The ice plate is characterized by mass density \( \rho_1 \), complex longitudinal plate wave velocity \( C_p \), vertical plate displacement \( w_1 \), mean thickness \( H \), over-ice roughness \( \eta \), and under-ice roughness \( \xi \). The air above and water below the ice plate are, respectively, characterized by densities \( \rho_a \), \( \rho_w \) and acoustic velocities \( C_{pa}, C_{pw} \). An incident underwater acoustic plane wave of velocity potential \( \varphi_0 \) in air, and vertical ice-plate displacement \( w_1 \), where \( \omega \) is acoustic frequency, \( K_w = \omega/C_{pw}, K_a = K_a(\alpha, \beta, \gamma) \), and \( \alpha, \beta, \gamma \) are the direction cosines of the incident wave. Thus \( \alpha^2 + \beta^2 + \gamma^2 = 1 \).

The three unknowns \( \varphi_0, w_1, w_1 \) are solved using three boundary conditions. The three boundary conditions are given by: (i) continuity of the normal velocity at \( z = (H/2) + \eta \):

\[
\hat{h}_u \cdot \nabla \varphi_0 = \hat{h}_w \cdot \frac{\partial w_1}{\partial t} \tag{1}
\]

(ii) continuity of the normal velocity at \( z = -(H/2) + \xi \):

\[
\hat{h}_w \cdot \nabla \varphi_0 = \hat{h}_w \cdot \frac{\partial w_1}{\partial t} \tag{2}
\]

and (iii) pressure discontinuity across the plate (see, e.g., [9]):

\[
H \rho_1 \frac{\partial^2 w_1}{\partial t^2} + (H^2 \rho_1 C_p^2/12) \nabla^4 w_1 + \rho_w \varphi_0 = 0, \quad \text{at} \ z = -H/2 + \xi \tag{3}
\]

where

\[
\varphi_0 = \varphi_{in} + \varphi_r, \quad \hat{h}_u = \left( -\eta_x, \ -\eta_y, \ 1 \right)/\sqrt{1 + \eta_x^2 + \eta_y^2}
\]

and

\[
\hat{h}_w = \left( \xi_x, \ \xi_y, \ -1 \right)/\sqrt{1 + \xi_x^2 + \xi_y^2}.
\]

The exact solution may be given in the generalized plane wave forms:

\[
\varphi_r(x, z, t) = e^{i\omega t} \int \exp \left[ -i(K_x \cdot x - K_{\omega z} z) \right] dG_1(\lambda, \mu_c)
\]

\[
\varphi_{in}(x, z, t) = e^{i\omega t} \int \exp \left[ -i(K_x \cdot x + K_{\omega z} z) \right] dG_1(\lambda, \mu_a) \tag{4}
\]

\[
w_1(x, t) = e^{i\omega t} \int \exp \left[ -iK_{1} \cdot x \right] dW(K_x, K_y) \tag{5}
\]

where \( x = (x, y), K_{1} = (K_x, K_y), K_x = K_w(\lambda, \mu_c), K_y = \omega/C_{pw}, K_{\omega z} = K_{\omega z}(\lambda, \mu_a), K_x, K_y \) represent the direction cosines of the reflected wave, and \( (\lambda, \mu_c, \mu_a) \) represent the direction cosines of the transmitted wave. Equations (4) or (5) satisfy the wave equation of the respective medium because \( \lambda_x^2 + \mu_x^2 + \nu_x^2 = 1 \) or \( \lambda_y^2 + \mu_y^2 + \nu_y^2 = 1 \). The unknowns are now represented by the generalized differential amplitudes \( dG_1, dG_2, \) and \( dW \). They can be found by substituting (4)–(6) into boundary conditions (1)–(3) and solving in the spectral space.
However, the exact solution is hard to implement. An approximate solution based on a small perturbation method of \[3\] and \[6\] is developed below. Accordingly, the boundary conditions are expanded in terms of the appropriate statistical boundary characteristics. The appropriate statistical boundary characteristics are the rms deviations,

\[ h'_a = \left( \frac{H}{2} + \eta \right)^2 \frac{1}{2} \]
\[ h'_w = \left( - \frac{H}{2} + \bar{\zeta} \right)^2 \frac{1}{2} \]
\[ S_a = \left( \frac{\partial \eta}{\partial x} \right)^2 \]
\[ S_w = \left( \frac{\partial \bar{\zeta}}{\partial x} \right)^2 \]

These perturbation parameters give the following expanded series for the unknown differential amplitudes:

\[ dG_\lambda(\lambda, \mu, r) = \sum_{m} \sum_{n} \sum_{p} \left( h'_a \right)^n \left( h'_w \right)^p S_a S_w dA_{mnpq}(\lambda, \mu, r) \]
\[ dG_\lambda(\lambda, \mu, \rho) = \sum_{m} \sum_{n} \sum_{p} \left( h'_a \right)^n \left( h'_w \right)^p S_a S_w dB_{mnpq}(\lambda, \mu, \rho) \]
\[ dW(K_x, K_y) = \sum_{m} \sum_{n} \sum_{p} \left( h'_a \right)^n \left( h'_w \right)^p S_a S_w dC_{mnpq}(K_x, K_y) \]

In the above context, unknowns are \( dA, dB, \) and \( dC. \) Assuming an acoustic wavelength that is long compared to the ice roughness and ice thickness, the expansion series in the above boundary conditions can be approximated by a truncation to the second order. This process produces three sets of three boundary conditions, one set for each perturbation order. They can be solved in the sequence of ascending perturbation orders for \( dA, dB, \) and \( dC. \) By substituting the results into (4) through (6), the three unknown fields are found to the second order in boundary characteristics.

For the scattering loss study in water, only the reflected field potential \( \varphi, \) is of interest. Accordingly, \( \varphi, \) takes the following form:

\[ \varphi(x, z, t) = e^{iut} \int \exp \left[ -i \left( K_r \cdot x - K_w \rho \varphi \right) \right] \left[ dA_{0000}(\lambda, \mu, \rho, \omega) \right] \]
\[ + (h'_a) dA_{1000}(\lambda, \mu, \rho, \omega) + (h'_w) dA_{0100}(\lambda, \mu, \rho, \omega) \]
\[ + (S_a) dA_{0010}(\lambda, \mu, \rho, \omega) + (S_w) dA_{0001}(\lambda, \mu, \rho, \omega) \]
\[ + (h'_a)^2 dA_{2000}(\lambda, \mu, \rho, \omega) + (h'_w)^2 dA_{0200}(\lambda, \mu, \rho, \omega) \]
\[ + (S_a)^2 dA_{0020}(\lambda, \mu, \rho, \omega) + (S_w)^2 dA_{0002}(\lambda, \mu, \rho, \omega) \]
\[ + (h'_a h'_w) dA_{1100}(\lambda, \mu, \rho, \omega) + (h'_a S_a) dA_{1010}(\lambda, \mu, \rho, \omega) \]
\[ + (h'_w S_w) dA_{0101}(\lambda, \mu, \rho, \omega) + (h'_a S_w) dA_{0110}(\lambda, \mu, \rho, \omega) \]
\[ + (h'_w S_a) dA_{0101}(\lambda, \mu, \rho, \omega) \]
\[ (7) \]

in which all \( dA \)’s (excepting \( dA_{0000} \)) are random and are found from the boundary conditions above.

An appropriate statistical reflected field characteristic is the reflected field spectrum \( \Pi(K_r). \) This is defined by

\[ \Pi(K_r) = \left( \frac{1}{2\pi} \right)^2 \int \exp \left( iK_r \cdot \xi \right) \varphi(x, z = 0, t) \varphi^*(x - \xi, z = 0, t) d\xi. \]
\[ (8) \]

The physical property of this reflected field spectrum is appreciated by the following relationship:

\[ \langle p_r^2 \rangle / \langle p_t^2 \rangle = \int \Pi(K_r) dK_r. \]
\[ (9) \]

Therefore \( \Pi(K_r) \) is the spectral (directional) contribution to the mean square pressure ratio of the reflected to the incident fields (intensity ratio). By carrying out the analysis indicated in (8), the following result is obtained:

\[ \Pi(K_r) = \left( R(\gamma) R^*(\gamma) \right) \]
\[ + \int K_r \left[ R(\gamma)(1 + iK_r Q_H) F_{KA}(k, k) \right] \Phi_k(k) dk \]
\[ + R^*(\gamma)(1 - iK_r Q_H) F_{KB}(k, k) \]
\[ + R^*(\gamma)(1 - iK_r Q_H) F_{KB}(k, k) \]
\[ + R^*(\gamma)(1 - iK_r Q_H) F_{KB}(k, k) \]
\[ + R^*(\gamma)(1 - iK_r Q_H) F_{KB}(k, k) \]
\[ + \left[ F_A(K_w - K_r) F_B(K_w - K_r) \Phi_A(K_w - K_r) \Phi_A(K_w - K_r) \right] + \left[ F_A(K_w - K_r) F_B(K_w - K_r) \Phi_A(K_w - K_r) \Phi_A(K_w - K_r) \right] \]
\[ + \Phi_A(K_w - K_r) \]
\[ (10) \]

where

\[ R(\gamma) = \left[ (\rho_p/\rho_w) - K_w f(\gamma)/(K_w \gamma) - iK_w f(\gamma) h(\gamma) / \omega^2 \rho_w \right] + \left[ (\rho_p/\rho_w) + K_w f(\gamma)/(K_w \gamma) - iK_w f(\gamma) h(\gamma) / \omega^2 \rho_w \right] \]
\[ (11) \]

reflection coefficient for the flat plate.

\[ f(\gamma) = \left[ 1 - (K_w / K_r)^2(1 - \gamma)^2 \right]^{1/2} \]
\[ h(\gamma) = -\rho_p \omega^2 + H^2 \rho_p C_p^2 K_w^2 (1 - \gamma)^2 \]
\[ (12) \]

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\[ \Phi_a(k) = \text{over-ice roughness spectrum}, \quad \Phi_w(k) = \text{under-ice roughness spectrum}, \quad \Phi_{AW}(k) = \text{over-} \quad \text{and under-ice roughness correlation spectrum}. \]

Functions \( F_{AA}, F_{BB}, F_{AB}, F_{BA}, F_A, \) and \( F_B \) are too involved to be presented here. However, they are independent of the roughnesses.

The scattering loss solution is given by the term weighted by \( b(K_m - K_r) \), the specular component in the reflected field. Based on this expression the propagation losses are predicted for various physical parametric values of the ice in the next section. The other part of (10) is the scattered component, which can be utilized for predicting the various spreading phenomena of the received signal. Results are presented in Section IV.

Before estimating the terms given by (10), the roughness spectra \( \Phi_A, \Phi_w, \) and \( \Phi_{AW} \) need to be specified. Information is very limited, especially for \( \Phi_A \) and \( \Phi_{AW} \). For under-ice roughness, Mellen [10] effectively proposed an isotropic two-dimensional spectrum:

\[
\Phi_w(k) = \frac{h_2}{\pi} \cdot \frac{k_0^2}{(k_0^2 + k^2)^2}
\]

where \( h_2 = 2 \text{ m} \) = rms under-ice roughness, \( k_0 = (2/44)(1/\text{m}) \), and \( k = |k| \).

The isotropic spectrum \( 2\pi k^2 \Phi_w(k) \), then, can be shown to have a maximum level at \( k = k_0/\sqrt{3} \) or a roughness containing a length scale of \( 3/k_0 \approx 38 \text{ m} \). Therefore the assumption of the validity of isotropy and certainly the stationarity with respect to the space implies that roughness encountered by the acoustic wave over long-range propagation has a statistical average property of isotropy at least over a patch size of an order of 38 m.

Although it is well known that the correlation between sails and keels is excellent and the rms roughness ratio between under-ice and over-ice is about 4, there is no information on \( \Phi_A \) and \( \Phi_{AW} \). For the purpose of estimating their effects on propagation loss,

\[
\Phi_a(k) = \frac{1}{16} \Phi_w(k) \quad \text{(15)}
\]

\[
\Phi_{AW}(k) = \frac{1}{4} \Phi_w(k) \quad \text{(16)}
\]

are utilized in the following computation unless otherwise specified. These forms reflect a roughness ratio of 4. As will be found later, the effect of these estimates is negligible due to an air backing; therefore the missing information is not critical.

III. THE PROPAGATION LOSS PREDICTIONS

The reflectivity \( |R(\gamma)|^2 \) contained in (10) is reproduced below.

\[
|R(\gamma)|^2 = |R(\gamma)|^2 + \int k 2 \text{Re} \left[ R(\gamma)(1 + iK_\gamma H) F_{AA}(k, k) \right] \Phi_a(k) dk + \int k 2 \text{Re} \left[ R(\gamma)(1 + iK_\gamma H) F_{BB} \Phi_w(k) \right] dk
\]

When both boundary roughnesses are absent, the reflectivity is given by \( R(\gamma) \) (equation (11)). This represents reflectivity for a flat thin ice plate. When both boundary roughnesses are present, scattering losses are estimated by integral terms. They contain roughness spectra \( \Phi_A, \Phi_w, \) and \( \Phi_{AW} \) and represent respective scattering losses due to over-ice, under-ice, and correlated roughnesses.

The estimate of the reflection coefficient based on (17) is properly identified as relating to the loss per bounce from the under-ice cover. Propagation experiments usually report results in dB loss per kyd. One such experiment is reported by [7]. Accordingly, propagation loss of 0.04813488 dB per kyd was empirically estimated for the 50-Hz acoustic wave of interest. From this information, dB loss per bounce can be estimated if the skip distance between two successive surface bounces is known. Depending on the dominant propagation mode, this distance and the associated incident angle at the surface can vary appreciably. Three representative sets of skip distances and surface angles are supplied by [11]. They are:

\[
\text{(skip distance in kyd, surface angle)} = (3.75, 6^\circ), (7, 8^\circ), (7.5, 9.25^\circ).
\]

Accordingly, dB loss per bounce and the corresponding reflection coefficient and incident angle are calculated to be,

\[
(1.94, 6^\circ), (0.979, 6^\circ), (0.962, 8^\circ), (0.959, 9.25^\circ).
\]

Although additional assumptions are required to obtain these values, the values can serve as benchmark points. Previously, Kuo [8] had shown the pressure release case to under-predict the loss per bounce by comparison to data for the same points. Recently, a Fast Field Program (FFP) capable of predicting propagation loss in dB/kyd was modified to incorporate the new reflection coefficient found above. Consistent results described below were obtained.

When both boundary roughnesses are absent, the reflection coefficient is for a flat thin ice plate; i.e., \( |R(\gamma)| = |R(\gamma)| \). It is given by (11). Fig. 2 displays this coefficient versus grazing angle for the previously studied soft boundary [7], [8] and thin plates of varying complex longitudinal plate wave velocity \( C_p \) as indicated. Because of air cover, the reflectivity magnitude is one for both soft boundary and plate of real \( C_p \); i.e., no plate absorption. This confirms the original thinking of previous investigators [7], [8], who treated the under-ice boundary as soft, implying that the thin ice plate is transparent to the low-frequency acoustic waves. However, the existence of ice plate absorption represented by the imaginary part of \( C_p \) reduces the reflectivity considerably at around a 25° grazing angle. If this were the angle of coincidence, the excited wave speed in ice should be at 1655 m/s (= 1500 m/s/cos 25°), which is much lower than the
plate wave speed of 3500 m/s and higher than subsonic flexural wave speed. However, it may be a shear wave speed. In the same figure, experimental data are also plotted. It can be seen that the flat thin plate theory under-predicts propagation loss up to plate absorption of a 20% loss tangent.

Fig. 3 presents the effect of changing the rough under-ice boundary condition from pressure release (soft) to that of the proposed thin loss-less plate (real $C_p$) theory. The proposed theory predicts considerably lower reflection coefficients at high grazing angles, but affects very little at the low grazing angles of interest. Varying the $C_p$ value from 3500 to 2457 m/s (not shown) does not significantly alter the above conclusion. Compared to experimental data, the loss-less rough thin plate theory predictions do not offer good agreement at small grazing incidences.

Fig. 4 shows the effect of changing the plate thickness from 4 to 6 m. For loss-less plate (real $C_p$), the effect on the reflection coefficient is felt only at high grazing angles. Again, the under-prediction of propagation loss at low grazing angles cannot be explained by the ice thickness.

However, a possible explanation of propagation loss can be given by synergistic effects among factors: roughness, thickness, and absorption. Fig. 3 also depicts the added effect of absorption to the rough plate theory. The added absorption effect brings down the reflection coefficient or increases propagation loss considerably at small grazing angles, but not at high grazing angles. Accordingly, the ice cover over the range of propagation experiments seems to have possessed an appreciable absorption capability in the range of a 10 to 20% loss tangent. At this time, there is no direct measurement of ice absorption to substantiate the above conclusion. However, the well-known fact about the ice cover being horizontally inhomogeneous with various flaws and mushy/slushy pockets cannot deny the above conclusion. Fig. 4 depicts the thickness effect of a rough-absorptive plate on the propagation loss estimate in grazing directions. Fig. 5 displays features of Fig. 3 in more detail.

In summary, the new theory either succeeded in predicting the measured loss or found a way by which higher propagation loss can be predicted. It is based on a simple and physically sound rough plate model. Accordingly, there are only a few important physical parameters: ice thickness, complex longitudinal ice plate wave velocity, and ice density. In contrast, if a full layer model were utilized, there would be a multitude of ice physical parameters. Then it would require a judgment as to which particular parameters were to be adjusted for the observed propagation loss. This process would not be unique in general.

The proposed rough ice plate theory includes the effects of the over-ice roughness and its correlation with under-ice roughness. Although an experimental data base is not available, reflection loss coefficients based on educated estimates of over-ice roughness and correlated over- and under-ice
specra given in Section II were computed. According to numerical computations, these effects are found to be negligible. This is probably caused by: (i) air cover of very small impedance, and (ii) smaller over-ice roughness than under-ice roughness.

IV. ARRIVAL-SIGNAL-SPREAD PREDICTIONS

The spread of a received signal in arrival time and angles can be caused by multipaths associated with off-specular or scattered components of the reflected field from a rough boundary. Additionally, if there are motions of source, receiver, or boundary, there will also be frequency spreads in the received signal. Predictions of the arrival time spread are made first, and the angular and frequency spreads are discussed secondly.

Fig. 6 depicts the scattering geometry of interest. Without loss of generality, a vertical plane x-z can be selected to contain a point source of coordinates \((-x_o, 0, -z_o)\), a point receiver of coordinates \((x_o, 0, -z_o)\), and the specular point of coordinates \((0, 0, 0)\). Spatial vectors \(r_{10}\) and \(r_{20}\) define the specular path, while spatial vectors \(r_1\) and \(r_2\) define an arbitrary off-specular path. The specular arrival time \(T_0\) and an arbitrary off-specular arrival time \(T\) are given by the following expressions:

\[
T_0 = \frac{(r_{10} + r_{20})}{C_w} \tag{18}
\]

\[
T = \frac{(r_1 + r_2)}{C_w} \tag{19}
\]

where

\[
r_{10} = |r_{10}| = \left[ x_o^2 + z_o^2 \right]^{1/2} \tag{20}
\]

\[
r_0 = |r_{20}| = \left[ x_o^2 + z_o^2 \right]^{1/2} \tag{21}
\]

\[
r_1 = |r_1| = \left[ (x + x_o)^2 + y^2 + z_o^2 \right]^{1/2} \tag{22}
\]

\[
r_2 = |r_2| = \left[ (x_0 - x)^2 + y^2 + z_o^2 \right]^{1/2} \tag{23}
\]

and \(C_w\) is a constant acoustic wave speed in water.

From the specular geometry, horizontal coordinates \(x_s\) and \(x_0\) can be expressed in terms of the source-receiver horizontal separation \(H_x\) and their depths, as in the following:

\[
x_s = H_x z_o/ (z_0 + z_o) \tag{24}
\]

\[
x_0 = H_x z_o/ (z_0 + z_o) \tag{25}
\]

For a given arrival time \(T > T_0\), the average received acoustic energy is the sum of all off-specularly reflected energy (with respect to the mean reflecting horizontal surface) from an entire elliptical ring situated on the x-y plane in Fig. 6. In the far field, a given point source can be regarded as a source of superposed plane waves emanating in all directions.

When one of these plane waves arrives at the randomly rough under-ice cover, it is scattered randomly. Statistically averaged, this reflected field was shown earlier by [6] to consist of one concentrated specular component and infinitely many diffused off-specular components described by (8) through (13). For brevity, the angular field spectrum \(\Pi(K)\) can be written in the following form:

\[
\Pi(K) = \Omega \delta(K_w - K_s) + \Gamma(K_w - K_s) \tag{26}
\]

where \(K_s = K_w(\lambda, \mu), K_w = K_w(\alpha, \beta), K_w = \) acoustic wave number in water, \((\lambda, \mu, \nu)\) = direction cosines (subscript \(r\) is dropped) of a reflected wave, \((\alpha, \beta, \gamma)\) = direction cosines of an incident wave, \(\delta = \) Dirac function,

\[
\Gamma(K_w - K_s) = F_s(K_w - K_s) F_s^*(K_w - K_s) \Phi_s(K_w - K_s) + F_s(K_w - K_s) F_s^*(K_w - K_s) \Phi_s(K_w - K_s) \tag{27}
\]

\[
\Phi_s = \text{over-ice roughness spectrum}, \Phi_w = \text{under-ice roughness spectrum}, \text{and } \Phi_{AW} = \text{over- and under-ice roughness correlation spectrum}. \Omega = \text{the specular component utilized for estimating the propagation loss in Section III. The off-specular component } \Gamma \text{ given by (27) has three contributions. They are over-ice, under-ice, and correlated roughness scattered contributions represented by terms weighted by } \Phi_s, \Phi_w, \text{ and } \Phi_{AW}, \text{ respectively. } F_s \text{ and } F_B \text{ are complicated functions (see details in Appendix A) of physical parameters characterizing incident and reflected waves and media. They are independent of boundary roughness represented by } \Phi_s, \Phi_w, \text{ and } \Phi_{AW}. \text{ The roughness dependence for the off-specular component } \Gamma \text{ is wavenumber selective at a wavenumber } (K_w - K_s), \text{ which depends on the scattering direction } \beta. \text{ This is in contrast to the specular component that depends on the entire wavenumber range of the roughness spectrum, because the specular energy loss is due to scattered energy loss in all directions.}

Therefore for a given arrival time \(T\) beyond the specular component arrival time \(T_0\), i.e., \(T > T_0\), the received intensity ratio \(IR\) at a point receiver is given by the off-specular part of (9) as in the following expression:

\[
IR = \iiint \frac{1}{r_1^2} \Gamma(K_w - K_s) \, dK_s. \tag{28}
\]

The factor \(1/r_1^2\) is to account for the geometrical spread from
the point source to an elementary scattering surface area. The integration domain, \(dK_r = K_r^2 d\lambda d\mu\), is over all scattering directions \((\lambda, \mu, \nu)\). A similar integral form was utilized by [12] for the ocean surface scattering.

For the time-spread investigation, it is more convenient to transform integration variables from \((\lambda, \mu)\) to \((T, \beta')\). \(T\) is the arrival time given by (19). The angle \(\beta'\) is the \(x\)-\(y\) plane polar angle at the specular point \((0, 0, 0)\) (see Fig. 6). The choice of this particular angle is to make correspondence between a scattering point \((x, y, 0)\) and a variable angle \(\beta'\) unique. The transformation is accomplished by two successive transformations indicated by the following Jacobians of the transformations:

\[
d\lambda d\mu = \frac{\delta(\lambda, \mu)}{\delta(x, y)} \frac{\delta(x, y)}{\delta(T, \beta')} dT d\beta'.
\]

(29)

The above Jacobians are derived in Appendix B. Accordingly, (28) is transformed into:

\[
IR = \int_0^{2\pi} \int_{T_1}^{T_2} \frac{\Gamma(K_w - K_r)}{r_1^2} \left| K_r^2 \right| \frac{R}{\delta T} dT d\beta'.
\]

(30)

The received level, 10 log\(_{10}\) IR, is computed as a function of the arrival time \(T\) and \(\beta'\), where \(T_1 = T_0 + (n - 0.5)\tau\), \(T_2 = T_0 + (n + 0.5)\tau\), \(\tau = \) pulse length = 10 wave periods, and \(n = 1, 2, 3, \ldots\). The numerical integration with respect to \(T\) and \(\beta'\) is carried out by IMSL software on a VAX computer. The results follow.

The first scattering geometry is defined by \(z_o = 90\) m, \(z_r = 137\) m, and \(H_o = 29000\) m. Accordingly, the specular arrival time is 1.9387 s. For a 0.2-s pulse of 50 Hz, the received levels are again affected very little by the amount of the absorption. Fig. 8 depicts the similar ice-thickness effects on the received levels as in Fig. 7; i.e., lower received levels for a thicker ice. This phenomenon is explained later by the angular spread. For this scattering geometry, 69.2 dB is the spread loss along the specular path of an incident angle of 85.5\(^\circ\). With reflection loss of \(-20\) log\(_{10}\) 0.98 = 0.18 dB (Section III), the expected specular level at 50 Hz is \(-69.38\) dB.

To investigate further the effect of ice thickness on received levels, a 25-Hz case is computed and depicted in Fig. 9. It shows higher received levels for a thicker ice. There seems to be a cross-over frequency below which the ice-thickness effect on a received signal level is reversed.

The scattering-induced multipath phenomenon causes not only the spread in signal arrival time but also the angular and frequency spreads of the received signal. For investigating the angular spread, integration variables in (28) should be transformed according to

\[
d\lambda d\mu = \left| \frac{\delta(\lambda, \mu)}{\delta(\cos \Theta_p, \Theta_s)} \right| d\cos \Theta_p d\Theta_s.
\]

From Appendix B, (28) becomes

\[
IR = \int \int \frac{\Gamma(K_w - K_r)}{r_1^2} \left| K_w^2 \right| |\nu| d\cos \Theta_p d\Theta_s
\]

(31)

where \(|\nu| d\cos \Theta_p d\Theta_s = |\nu| d\Omega = 4\pi |\nu|^2 (dx/dy) r_1^2\) and \(\Omega = \) solid angle extended at the receiver. Therefore \(\Gamma(K_w - K_r)K_w^2 |\nu| /r_1^2\) is the intensity ratio per unit solid angle, and \(\Gamma(K_w - K_r)K_w^2 |\nu|^2 /r_1^2\) is the scattering strength in the direction of \((\Theta_p, \Theta_s)\) at the point receiver.

For every given pair of \((\Theta_p, \Theta_s)\), calculations are made for relative received (dB) levels,

\[
dB = 10\log_{10} \left[ \frac{\Gamma(K_w - K_r)}{r_1^2} K_w^2 |\nu| \right]
\]

(32)
Fig. 9. Time spread of the received signal versus time of arrival, where f is:

\[ T = \left( r_1 + r_2 \right) / C_w \]  

(33)

and the Doppler \( Q = (f - f_0) / f_0 \) (see, e.g., [13]),

\[ Q = \frac{C_w - V_T}{C_u} - \frac{C_w - V_R}{C_u - V_s} = 1 \]  

(34)

where \( f \) is received frequency, \( f_0 \) is source frequency, \( V_p \) = \( v_x + v_y + v_z \) component of the source velocity: \( v_i = (v_x, v_y, v_z) \), \( V_R = v_R x + v_R y + v_R z \) = \( r_1 \) component of the receiver velocity: \( v_R = (v_R x, v_R y, v_R z) \), \( V_p = v_B x + v_B y + v_B z \) = \( r_1 \) component of the boundary velocity: \( v_B = (v_B x, v_B y, v_B z) \), and \( V_T = v_T x + v_T y + v_T z \) = \( r_2 \) component of the boundary velocity \( v_B \) (see Fig. 10). The results are presented in the following paragraphs.

For the scattering geometry of Fig. 6, selected numerical values are: \( z_s = 4000 \text{ m}, \ z_0 = 4000 \text{ m}, \) and \( H_s = 22000 \text{ m}. \) For this scattering geometry [14], the specular direction is given by (\( \Theta_s, \Phi_s \)) = (70°, 180°), the specular arrival time is 15.6063 s, and the spreading loss for the specular path is 87.39 \( \Delta B \). Computations of \( DB, T, \) and \( Q \) according to (32)–(34) are carried out for every 1° around the specular direction (70°, 180°) over 10° on both sides. Different cases are investigated and reported in the following for the same scattering geometry.

The first case is defined as follows: acoustic frequency = 50 Hz, \( C_p = 3500 \text{ m/s}, \) \( H = 4 \text{ m}, \) \( V_{1x} = 12.24 \text{ m/s} = 24 \text{ kn}, \) and other velocities are 0. Fig. 11 maps the relative off-specular level per unit solid angle in an angular field. The maximum off-specular level at (\( \Theta_p, \Theta_s \)) = (74°, 180°) is 0.67 dB higher than the scattering level from the specular direction (\( \Theta_p, \Theta_s \)) = (70°, 180°). The difference in the polar angle direction of 4° was defined by [14] as the localization error. Their predictions were based on simple geometric acoustic reflections from the illuminated facets in the rough boundary. The dB levels in Fig. 11 are only relative and can be compared to neither the dB levels of the specular component nor the received signal at different times. The dB levels here measure the relative scattering strength in the angular field, while the other dB levels refer to a unit source level at a unit distant from the source. Fig. 11 addresses only the off-specular or the scattered field. In the specular direction there will be another acoustic energy contribution from the concentrated specular component. The specular component level will be much above that of the off-specular component so that there should not be any problem detecting the specular direction. Two obvious reasons for low off-specular or scattered component level are (i) the scattered component is difused while the specular component is concentrated, and (ii) the law of geometrical spread for the scattered component follows \( 1/(r_0 + r_0)^2 \), while that for the specular component follows \( 1/(r_1 + r_1)^2 \). This is in contrast to the high-frequency scattering field studied in [14]. Although their high-frequency scattering process is based on summing many locally specular reflections from the entire rough boundary, the results show no specular component with respect to the source–mean boundary–receiver geometry.

Fig. 12 depicts the corresponding iso-arrival-time contours. Superposition of Figs. 11 and 12 indicates that the scattered component arrives earliest from a specular direction at 15.606 s, while the maximum-level scattered component arrives later at 15.7 s. It is seen that the iso-arrival-time
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Fig. 12. Angular spread time (s) contour; 50 Hz; $C_p = (3500$ m/s, 0); $H = 4$ m; and $V_{ex} = 12.24$ m/s.

Fig. 13. Angular spread Doppler $Q$ contour; 50 Hz; $C_p = (3500$ m/s, 0); $H = 4$ m; and $V_{ex} = 12.24$ m/s.

Fig. 14. Angular spread Doppler $Q$ contour; 50 Hz; $C_p = (3500$ m/s, 0); $H = 4$ m; $V_{ex} = 12.24$ m/s; and $V_{ex} = -12.24$ m/s.

The contour forms ellipses on the $\Theta_p - \Theta_a$ plane and that their centers shift in the positive x-direction as the arrival time increases. Fig. 13 shows the corresponding $Q$ contour as a result of source motion in the positive x-direction at 12.24 m/s (24 kn). Similar relative $Q$ level distributions (i.e., there exists a maximum level $Q$) presented by [14] can be obtained along a given $\Theta_a$ by superimposing Figs. 11 and 13. In this case, the level distribution of $Q$ is skewed.

The second case (not shown) is the same as the first except for the value $C_p = (3500, 700$ m/s). Despite ice absorption represented by the imaginary part of $C_p$, the received scattered level contours change very little. At a frequency of 50 Hz, the specular reflectivity around the incident angle range of 60° to 80° (or 10° to 30° grazing angles) is quite high at about 0.94. Therefore there is only a small portion of the incident energy which is available for the scattered field and ice plate absorption. After scattering diffusion, the effect of the small plate absorption on a scattered field is seen to be small.

The third case is the same as the first case except for the added motion of the receiver. Both source and receiver are moving toward each other at 12.24 m/s (24 kn). Level and time contours are given in Figs. 11 and 12 as before. The new $Q$ contours are given in Fig. 14. Contours are seen to form concentric ellipses about the specular point. Compared to those of Fig. 13, $Q$ values are doubled, as they should be. Superposition of Figs. 14 and 11 indicates the relative level distributions of $Q$ to be dependent on an outward direction from the specular point. Except for directions near $\Theta_p = 180°$ and $\Theta_a \geq 70°$, the level decreases monotonically as $Q$ decreases. Superpositions of Fig. 12 with Figs. 13 and 14 indicate that the $Q$ spectrum spreads in time. Near the specular arrival time, the $Q$ spectrum is quite narrow and then spreads to contain more and more different $Q$ values as time of arrival progresses. For the situation presented in Fig. 14, however, this spread is more limited by the nature of the $Q$ contour near the specular point.

The last case is the same as the first case except for the 6-m ice plate thickness. The results are shown in Fig. 15. As previously shown, the specular energy loss near the incident angle of 70° or for small grazing angles of 20° was negligibly affected by the thickness of the loss-less plate. However, the energy diffusion process by scattering can be noticeably affected by an increase in the plate thickness, which tends to increase the plate rigidity. Also, the incident angles of off-specular components are not limited to the specular angle. A comparison of Figs. 15 and 11 indicates that the thicker plate spreads the scattered energy to a wider angular region. The same process tends to reduce the received time-spread levels shown in Figs. 7 and 8 because of the lower scattered energy.
density in the same angular region within the same time period of arrival indicated in Fig. 12.

V. RESULTS AND CONCLUSIONS

Estimates of reflected field loss and spreading phenomena due to roughness scattering were made by utilizing a model of a thin ice plate that was rough and lossy. The method of analysis was based on Marsh’s [3] and Kuo’s [6] perturbation model. The model naturally divided the reflected field into specular and off-specular components. The selected acoustic frequency was 50 Hz. The under-ice roughness spectrum was empirically derived by [10]. The forward propagation loss was then estimated by the specular component. The results of varying the parameters of complex longitudinal plate velocity, ice thickness, and angle of incidence improved the propagation loss estimates when compared to the experimental data. The study suggested the importance of the combined effects of roughness scattering, ice absorption, and ice plate thickness. If the mean ice thickness was taken to be about 4 m and Mellens’s under-ice roughness spectrum [10] was assumed valid, then the theory predicted the ice loss tangent to be in the range of 10 to 20%.

It was also found that over-ice roughness and its correlation with under-ice roughness were not important when the ice cover was overlain by the air.

The off-specular or scattered component of the reflected field was applied to a point source and point receiver geometry for a study of various spreading phenomena. Levels of the time-spread signal were found to depend significantly on the ice thickness but negligibly on the ice absorption. Angular spread levels of the scattered field depended little on the ice absorption but noticeably on the ice thickness. Thicker ice tended to spread scattered energy in the wider angular region. The maximum angular spread signal level was found to arrive from an off-specular direction. According to previous investigators [14], this direction changed as the boundary roughness changed. Features of iso-Q contours were found to change considerably with the relative motion vectors of source and receiver, and Q or the Doppler shift spectrum was predicted to spread with time of arrivals. It can be concluded that time/angular/frequency spreads for under-ice cover are qualitatively similar to those from other roughness boundaries.

APPENDIX A

DETAILS ON COEFFICIENTS IN EQUATION (27)

\[ F_A(K_w - K_r) = -ik_\gamma \left[ 1 - R(\gamma) \left[ f(\nu) - f(\gamma) \right] T(\nu)/2\nu \right] + iK_w(K_w - K_r)Q(\gamma)T(\nu)/(2\nu K_w) \]
\[ - K_w^2 H(\rho_d/\rho_w)(K_w/K_w)f^2(\gamma) \]
\[ \cdot \left[ f(\nu) - f(\gamma) \right] Q(\gamma)Q(\nu)/(4\nu) \]
\[ + K_w^2 H(\rho_d/\rho_w)(\gamma/\nu)[1 - R(\gamma)] \]
\[ \cdot Q(\nu)f(\gamma) \left[ f(\nu) - f(\gamma) \right]/4 \]
\[ - HK_w^2(\rho_d/\rho_w)f(\gamma) \left[ f(\nu) - f(\gamma) \right] \]
\[ \cdot \gamma \left[ Q(\gamma)Q(\nu)/(4\nu) \right] \]
\[ + HK_w[K_w(K_w - K_r)] \]
\[ \cdot \left[ f(\nu) \left[ (\omega^2\rho_d - iK_w/\nu) \right] \right] \]
\[ \cdot Q(\nu)/\nu \left[ 2K_w(\omega^2\rho_d) - 1 \right] \gamma \left[ Q(\nu)/(2\nu) \right] \]
\[ - HK_w[K_w(K_w - K_r)] \gamma \left[ Q(\gamma)T(\nu)/4 \right] \]
\[\begin{align*}
+ HK_w (K_{w_j} - K_{r_j}) \gamma [1 + R(\gamma)] R(\gamma)/(4\pi)
+ HK_w (K_{w_j} - K_{r_j}) [1 + R(\gamma)] [1 + R(\nu)]/4
\end{align*}\]  
(A2)

where

\[f(\gamma) = \left[1 - (K_{w_j}/K_{r_j})^2 (1 - \gamma^2)\right]^{1/2}\]
\[R(\gamma) = \left[\left(\rho_{w}/\rho_u - K_{r_j} f(\gamma)/(K_{w_j})\right) - iK_{r_j} f(\gamma)/(K_{w_j})\right]^{1/2}\]
\[Q(\gamma) = \left[(\rho_u/\rho_u) T(\gamma) - iK_{r_j} f(\gamma)/(K_{w_j})\right]^{1/2}\]
\[h(\gamma) = -H_{r_j} \rho_{w}^2 + H_{s}^2 C_{r_j} K_{w_j}(1 - \gamma^2)^2/12 + \rho_{w}^2 \theta\]
\[T(\gamma) = 2(\rho_u/\rho_u) \left[(\rho_u/\rho_u) + K_{r_j} f(\gamma)/(K_{w_j})\right]^{1/2}\]
\[K_{w_j} = K_{w},
K_{r_j} = K_{r}.
\]

**APPENDIX B**

**JACOBIANS OF TRANSFORMATIONS**

To obtain (\(\lambda, \mu\)) as a function of (\(x, y\)):

From Fig. 6, incident and reflecting vectors are defined by

\[r_1 = (x + x_s) \hat{t} + y \hat{j} + z \hat{k}\]
\[r_2 = (x_0 + x) \hat{t} - y \hat{j} - z \hat{k}\]  
(B1)

Then the following relationships can be obtained:

\[\hat{t} \cdot r_1 = r_1 \alpha = x + x_s\]
\[\hat{j} \cdot r_1 = r_1 \beta = y\]
\[\hat{k} \cdot r_1 = r_1 \gamma = z_s\]

and

\[\hat{t} \cdot r_2 = r_2 \lambda = x_0 - x\]
\[\hat{j} \cdot r_2 = r_2 \mu = -y\]
\[\hat{k} \cdot r_2 = r_2 \nu = -z_0\]

where

\[r_1^2 = (x + x_s)^2 + y^2 + z_s^2\]
\[r_2^2 = (x_0 - x)^2 + y^2 + z_0^2\]  
(B3)

From the above, (\(\alpha, \beta, \gamma\)) and (\(\lambda, \mu, \nu\)) can be solved in terms of (\(x, y\)). They are:

\[(\alpha, \beta, \gamma) = \left(\frac{x + x_s}{r_1}, \frac{y}{r_1}, \frac{z_s}{r_1}\right)\]
\[(\lambda, \mu, \nu) = \left(\frac{x_0 - x}{r_2}, -\frac{y}{r_2}, -\frac{z_0}{r_2}\right)\]  
(B5)

The following Jacobian, then, can be obtained:

\[
\frac{\partial (\lambda, \mu)}{\partial (x, y)} = \begin{vmatrix}
\frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} \\
\frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial y}
\end{vmatrix} = \frac{z_0^2/r_2^2}{z_0^2/r_2^2}.\]  
(B7)

To obtain (\(x, y\)) as a function of (\(T, \beta'\)):

From Fig. 6, the following trigonometric identities can be obtained:

\[R_x^2 = R^2 + x_0^2 - 2R_{x_0} \cos \beta'\]
\[R^2 = R_0^2 + x_0^2 - 2R x_0 \cos \beta\]  
(B8)

They can be solved for (\(R_x, \cos \Theta\)) in terms of (\(R, \beta'\)). The results are (B8) and

\[\cos \Theta = (x_0 - R \cos \beta')/R_x.\]  
(B9)

Since

\[C_u T = x_0 + r_1\]
\[r_1^2 = r_1^2 + z_s^2\]
\[r_2^2 = r_2^2 + z_0^2\]
\[R_x^2 = R_0^2 + (x_0 + x_s)^2 - 2R_0(x_0 + x_s) \cos \Theta\]

then expression can be obtained:

\[C_u^2 T^4 + \left[(x_0 + x_s)^2 - 2R_0(x_0 + x_s) \cos \Theta + z_s^2 - z_0^2\right]^2
- 2C_u^2 T^2[2R_0^2 + (x_0 + x_s)^2]
- 2R_0(x_0 + x_s) \cos \Theta + z_s^2 + z_0^2 = 0.\]

By utilizing (B8) and (B9), \(R_x\) and \(\cos \Theta\) can be eliminated. The resulting expression can then be rearranged to obtain a quadratic equation of \(R_x\):

\[aR_x^2 - bR_x - c = 0\]

where

\[a = 4(C_u^2 T^2 - H_0^2 \cos^2 \beta') > 0\]
\[b = 4H_0 C_u^2 \frac{z_0 - z_s}{z_0 + z_s} (T^2 - T_0^2) \cos \beta'\]
\[c = C_u^2 (T^2 - T_0^2)[T^2 - T_0^2(z_0 - z_s)^2/(z_0 + z_s)] > 0.\]

The physically possible root of the quadratic equation is

\[R_x(T, \beta') = \frac{b + \sqrt{b^2 + 4ac}}{2a}\]

\[= (C_u^2/2)\left[H_0(z_0 - z_s)/(z_0 + z_s)\right]
\cdot (T^2 - T_0^2) \cos \beta' + T(T^2 - T_0^2)^{1/2}\]
\cdot \left[C_u^2 T^2 - H_0^2 \cos^2 \beta' - (z_0 - z_s)^2/\right]
\cdot \left(z_0 + z_s\right)^2(C_u^2 T_0^2 - H_0^2 \cos^2 \beta')^{1/2}/
\cdot \left(C_u^2 T^2 - H_0^2 \cos^2 \beta'\right).\]  
(B10)

Now it is simple to obtain \((x, y)\) in terms of \((T, \beta')\). From Fig. 6,

\[x = R(T, \beta') \cos \beta'\]  
(B11)
\[y = R(T, \beta') \sin \beta'.\]  
(B12)
Therefore,
\[
\frac{\partial (x, y)}{\partial (T, \beta')} = \left| \begin{array}{cc}
\frac{\partial x}{\partial T} & \frac{\partial y}{\partial T} \\
\frac{\partial x}{\partial \beta'} & \frac{\partial y}{\partial \beta'}
\end{array} \right| = \left| \frac{\partial R}{\partial T} \right|
\]

(B13)

where \(\partial R/\partial T\) can be obtained from (B10); that is,
\[
\frac{\partial R}{\partial T} = \frac{C_m^2}{2} \left[ C_n^2 T^2 \left( T_2^2 S^2 + C_n^2 T^2 \left( T^2 - T_2^2 \right) \right) + 2 H_0 (z_0 - z_2) \left( z_0 + z_2 \right) \right.
\]
\[
\left. \left( T \cos \beta' \sqrt{T^2 - T_2^2} \sqrt{S^2} \right) - H_0^2 \cos^2 \beta' \left( 2 H_0 (z_0 - z_2) \left( z_0 + z_2 \right) \right) \right.
\]
\[
\left. \left( T \cos \beta' \sqrt{T^2 - T_2^2} \sqrt{S^2} \right) + (2T^2 - T_2^2) S^2 + C_n^2 T^2 \left( T^2 - T_2^2 \right) \right] / \left( C_n^2 T^2 - H_0^2 \cos^2 \beta' \right)
\]

(B14)

where
\[ S^2 = C_n^2 T^2 - H_0^2 \cos^2 \beta' - (z_0 - z_2)^2 / \left( z_0 + z_2 \right)^2 \]

To obtain \((\lambda, \mu)\) as a function of \((\cos \Theta_p, \Theta_a)\):

Polar angle \(\Theta_p\) and azimuthal angle \(\Theta_a\) are defined in Fig. 6 at the receiver. From this definition and (B6), the following expressions can be obtained:
\[
\lambda = -\sin \Theta_p \cos \Theta_a = (x_0 - x) / r_2
\]
(B15)
\[
\mu = -\sin \Theta_p \sin \Theta_p = -y / r_2
\]
(B16)
\[
\nu = -\cos \Theta_p = -z_0 / r_2.
\]
(B17)

Then the following can be obtained:
\[
\frac{\partial (\lambda, \mu)}{\partial (\cos \Theta_p, \Theta_a)} = \left| \nu \right|
\]

(B18)
\[
\frac{\lambda}{\nu} = x_0 + \frac{\lambda}{\nu} = \frac{x}{x_0} = \frac{y}{y_0}
\]

(B19)
\[
\frac{y_0}{\nu} = \frac{y}{y_0} = \frac{y_0}{\nu}
\]

(B20)

This implies that once \((\Theta_p, \Theta_a)\) are given, \((\lambda, \mu, \nu), (x, y), \) and \((\alpha, \beta, \gamma)\) can be obtained in sequence by also utilizing (B5).

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