Parameter Compatibility Relations for Accelerated Testing

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Key Words — Accelerated testing, time transformation, parameter change with stress level

Reader Aids —  
Purpose: Tutorial, widen state-of-the-art  
Special math needed for explanations: Statistical theory  
Special math needed to use results: Same  
Results useful to: Accelerated testing practitioners, reliability theoreticians

Summary & Conclusions — Use of acceleration studies implies that stress does not change the mechanisms of failure but only contracts the time to reach failure level. This implication is translated into a requirement that the parameter set for the failure distribution does not change, but values of the parameters may. The generality of the relationship between parameter values at various stress levels and the time transformation that connects the respective distributions is emphasized. These are termed compatibility relations. An example of compatibility relations for acceleration studies described by a Weibull distribution is given in detail. Also, the compatibility relations between the coefficients in the time transformation in acceleration studies based on exponential, gamma, lognormal, and Weibull distributions and respective parameter values are discussed generally. Compatibility relations can be used to: 1) Improve estimation procedures, 2) Check the consistency of parameter estimations during analysis of acceleration test data, and 3) Check the validity of the use of acceleration studies in the first place.

1. INTRODUCTION

Although some efforts have been made to use non-parametric techniques [1, section 9.4.1] that require measurements at normal operating conditions, typically one models the effect of stress through the changes of parameters of distributions for failure statistics. Acceleration studies involve at least two measurements of failure statistics under higher stresses than normal operating conditions; they are used to estimate failure time distributions at the normal operating conditions. The validity of the accelerated test is then based on the assumption that a given parameter set does not change but rather only values of the parameters can change with stress. The change in parameter values is accompanied by a change in time to reach a given failure level. Then there must be a relationship between parameter values and the structure of the time transformation inherent in the acceleration study. We call such relationships compatibility relations; they have already been introduced elsewhere [2, 3]. Our objectives are to emphasize further their existence and usefulness.

The next section summarizes the notation. Compatibility relations and their application in a well-studied example and more generally to exponential, gamma, lognormal and Weibull distributions are discussed in the third section.

2. NOTATION

\[ x, y, t \]  
Failure times at various stress levels  
\( \kappa \)  
general parameter set; brackets indicate \( \kappa \) can represent more than one parameter  
\( \eta \)  
location parameter  
\( \sigma \)  
scale parameter (must be positive)  
\( \beta \)  
shape parameter (must be positive)  
\( h\{...,k\} \)  
hazard rate with parameter set \( \{...,k\} \)

The four distributions most commonly used in acceleration studies are: exponential, Weibull, gamma, lognormal. The Cdf of each is given here for 3 parameters: location, scale, shape.

\[
\text{Weib}_\text{Cdf}(x; \eta_x, \sigma_x, \beta_x) = 1 - \exp\left(-\left(\frac{x-\eta_x}{\sigma_x}\right)^{\beta_x}\right), \ x > \eta_x, \ \sigma_x > 0, \ \beta_x > 0
\]

\[
\text{Expo}_\text{Cdf}(x; \nu_x, \nu_x) = \text{Weib}_\text{Cdf}(x; \eta_x, \sigma_x, 1)
\]

\[
\text{Gamm}_\text{Cdf}(x; \eta_x, \sigma_x, \beta_x) = \frac{\sigma_x^{-\beta_x}}{\Gamma(\beta_x)} \int_{\eta_x}^{x} (u-\eta_x)^{\beta_x-1} \exp\left(-\frac{u-\eta_x}{\sigma_x}\right) du,
\]

\[ x > \eta_x, \ \sigma_x > 0, \ \beta_x > 0 \]

\[
\text{LogN}_\text{Cdf}(x; \eta_x, \sigma_x, \beta_x) = \frac{\beta_x}{(2\pi)^{1/2}} \int_{\eta_x}^{x} \left( \frac{1}{u-\eta_x} \right) \exp\left(-\frac{1}{2} \beta_x \log\left(\frac{u-\eta_x}{\sigma_x}\right)^2\right) du,
\]

\[ x > \eta_x, \ \sigma_x > 0, \ \beta_x > 0 \]

3. COMPATIBILITY CONDITIONS  
IN ACCELERATED TESTING

Accelerated testing involves: a) Measurements of failure statistics for at least two elevated stress levels, and b) Extrapolation back to normal operating conditions. However, any pair of time dependent Cdf's can be used to define a one-to-one relation between times to reach given Cdf levels. Thus, there is,
in principle, no relationship between the failure mechanisms that caused the individual Cdfs to reach that same level. But if accelerated testing is to be valid, the times to reach a given failure level must not only be in a one-to-one relation, the failure mechanism must be the same at each stress, including the normal stress. The idea that the failure mechanism must be the same everywhere in an acceleration study leads to the requirement that distributions at various stress levels can have different parameter values, but must all share the same parameter set.

Tobias & Trindade [2, chapter 7] have most recently shown that: a) The shape parameters for Weibull and lognormal distributions must be the same if the failure times at various stress levels are related by a constant scale factor, b) In a specific example for a Weibull distribution, the time scale factor is given by an appropriate ratio of the Weibull scale parameters at two different stress levels. Cox & Oakes [3] have discussed accelerated life model in a general way that includes parameter compatibility relations of the type discussed here.

Our aim here is to re-emphasize the self-consistency requirements between the relationship of failure times and the relationship of the parameters in the distribution used to describe the failures at different stress levels. A first step in the analysis of failure data should be to determine the nature of the time transformation associated with the acceleration due to change in stress level. This can be done either directly by a regression of times at equivalent stages of the failure Cdf at various stress levels, or indirectly by using compatibility relations to test a hypothesis about the form of the time transformation. Once established, the time transformation imposes self-consistency conditions on estimation procedures for parameters of the failure distribution by means of the compatibility relations. When subjected to the constraints of compatibility relations, estimation procedures provide improved parameter estimates — because information about the relationship among tests at various level is incorporated into the analysis. This contrasts with procedures in which parameters are estimated independently at each stress level.

These points are illustrated by considering a set of 3 Weibull distributions with failure times, t, x, y — all having: a) the same time origin, but b) different, fixed stress levels. In general, the scale and shape parameters are different for different stress levels. However, the failure process reaches the same level in all the tests when the Cdfs are equal. This occurs when

\[(t/a_1)^{\beta_x} = (x/a_1)^{\beta_x} = (y/a_1)^{\beta_y}\]  

(1)

Then, for example, the failure times y and x are related by

\[y = a_y(x/a_1)^{\beta_y/\beta_y}\]  

(2)

It is clear that y can be linearly related to x if \(\beta_x = \beta_y\). When that is true for our chosen order of stress levels, the acceleration factor between y and x is \(a_y/a_1\). Thus, a scale transformation in failure times for a Weibull distribution requires the shape parameter to be fixed and the scale factor to be the appropriate ratio of the Weibull scale parameters, and vice versa. Estimation of parameters must be constrained to satisfy such compatibility relations for self-consistency in the interpretation of stress response as accelerated testing.

Figure 1 shows three Weibull Cdfs with equal origins and shape factors so they are mutually related by time transformations that are just changes of scale. The relation between y and x, for example, can be obtained directly by tabulating the values of \(a_y, a_x, \beta_y, \beta_x\) for a series of equal values of the Cdfs.

(Data points are from [3].)

Figure 1. Weibull Distributions at three Stress Levels.

The data points in figure 1 correspond to those given by Tobias & Trindade [2, p 121, table 7.2] for three sets of 40 capacitors tested at 85 C, 105 C, 125 C with readout times at 24, 72, 168, 300, 500, 750, 1000, 1250, 1500 hours. We have plotted the interpolation of the data in figure 1 on an expanded scale and tabulated values of \(x\) and \(y\) for which the corresponding Cdfs are equal. We say "Cdf" because in such a procedure no specification is made concerning the appropriate failure distribution. Figure 2 is a plot of the paired values of \(x\) for 105 C and \(y\) for 125 C and their linear regression. The relatively small value of the \(x\) intercept indicates that the assumption of a scale change in time is appropriate. The acceleration factor from 105 C to 125 C is estimated to be 9.24 by this means.

In general, use of raw data in this procedure reduces the number of allowable comparison points for the two times. For example, here there are only 6 points in the comparison while each original data set had 9 readout times.

It is even easier to check the assumption of a scale change in time if one specifies a failure distribution and uses a compatibility relation. In this case the specification of a Weibull distribution means that one should obtain estimates of the shape parameters that are approximately equal. A least squares fit to —
The regression of failure times at 2 stress levels yields the data in column #1 of table 1. The acceleration factor from 105°C to 125°C is estimated to be 9.12 by this means—compared to 9.24 above. (See table 1, note 5 about uncertainties in the estimates.)

Now observe that the closer the Cdf is to 1 the more accurate the shape factor. The fact that the point estimates of the shape factors at the two highest temperatures are close to one another and, in this case, coincide to 2 significant figures indicates to us that an assumption that the acceleration is represented by a change of time scale is appropriate. However, given that the shape factors must, for self-consistency, all be equal for acceleration represented by a scale change, we must constrain the graphical estimates to all having the same shape factor. Based on the fact that data at the highest temperature provide the most information about the shape parameter, we chose the shape parameter to be 0.71 for all three tests. The results from this analysis are shown in column #2 of table 1. The solid curves in figure 1 correspond to the Cdfs with shape parameter 0.71, and scale parameters 17630, 2180, 239 hours. The fit to the data in figure 1 compares favorably with a similar fit of the same data with the parameters obtained by Tobias & Trindade with a sophisticated MLE where the shape parameters were constrained to be equal.

However, our constrained graphical technique here appears to be somewhat less successful in fitting the data to the Arrhenius model:

\[
\sigma = \sigma_0 \exp \Delta H / k T
\]

(Units are ignored in taking logs of the shape parameter)

Our values (from figure 3) are shown in column #3 of table 1, and contrasted with those of Tobias & Trindade [2, p 145, table 7.7] in column #4 of table 1.

The magnitude sign \( | \) signifies that units "hours" are ignored in taking the logarithm.

Figure 3. Arrhenius Plot of log(Scale Parameter) vs \( 1/(kT) \)

<table>
<thead>
<tr>
<th>Temperature</th>
<th>(col #1)</th>
<th>(col #2)</th>
<th>(col #3)</th>
<th>(col #4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \theta )</td>
<td>( \beta = 0.71 )</td>
<td>( \beta = 0.71 )</td>
<td>( \beta = 0.70 )</td>
</tr>
<tr>
<td>85°C</td>
<td>38780</td>
<td>0.71</td>
<td>17630</td>
<td>18550</td>
</tr>
<tr>
<td>105°C</td>
<td>2180</td>
<td>0.71</td>
<td>2180</td>
<td>1940</td>
</tr>
<tr>
<td>125°C</td>
<td>239</td>
<td>0.71</td>
<td>239</td>
<td>254</td>
</tr>
<tr>
<td>( \theta )</td>
<td></td>
<td>1.32</td>
<td></td>
<td>1.20</td>
</tr>
<tr>
<td>( \delta )</td>
<td></td>
<td>4.95 \times 10^{-15}</td>
<td></td>
<td>1.51 \times 10^{-13}</td>
</tr>
</tbody>
</table>

Units of \( \theta \)'s are hours  
Units of \( E \) are eV

Notes:

1. Column #1: The least-squares regression estimated the scale & shape parameters for each data set.
2. Column #2: The shape parameter was fixed at 0.71 for all data sets. The least-squares regression estimated only the scale parameter for each data set.
3. Column #3: The shape parameter was fixed at 0.71. The other parameter were estimated from the data.
4. Column #4: These data are MLEs from [2].
5. Uncertainty of the estimates: No interval estimates were made for any of the calculations. Thus the significant figures shown do not necessarily reflect any of the uncertainties in any of the calculations.
The constrained graphical estimation is a major improvement over the independent graphical estimation but somewhat less satisfactory than the constrained MLE in column #2 of table 1. The point to be emphasized, however, is that whatever estimation procedure one uses, one should impose the appropriate compatibility relations as constraints to be self-consistent, and to obtain the most trustworthy results that the estimation procedure can yield.

With this example in hand, let us discuss compatibility relations in a more general context. Consider the failure statistics for like sets of components subjected to different levels of stress. A particular level of cumulative failure is reached at different times inversely ordered to the ordering of stress levels. With the accelerated testing condition that parameter values may change but the parameter set does not change with change of stress, we write:

\[ y = \left( \frac{\sigma_y}{\sigma_x} \right) x; \beta_x = \beta_y, \eta_x = \eta_y = 0 \]  

Eq. (11) incorporates the simplest and most common compatibility relation:

The coefficient in the time scale transformation is the ratio of the scale factors, and the shape factors are equal. When the shape parameters are not equal, a form of (9) is:

\[ y = \left( \frac{\sigma_y}{\sigma_x} \right)^{\frac{x}{\eta_x}} ; \eta_x = \eta_y = 0 \]  

Then the time transformation is nonlinear, but one must still deal with compatibility relations.

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REFERENCES


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