Abstract

A new and rigorous analysis is developed for treating a common problem in hardness assurance where all the parts fail between two consecutive test levels in a step-stress test. The present method, which is the mathematically correct way to calculate confidence and probability of part survival, gives significantly more precise and conservative estimates of survivability than do previous treatments of the same problem. The analysis also yields a methodology which encompasses LTPD testing, overtesting and suggests a new kind of overtest using two overtest levels.

1. INTRODUCTION

In step-stress testing, the stress is increased in discrete steps and the number of failed parts is recorded for each level. To reduce the amount of testing, the stress levels are often widely spaced and as a result, sometimes all the parts fail abruptly between two consecutive radiation levels (a single "bin"). In such cases, rigorous derivations of a mean and standard deviation (s.d.) for use in a survivability calculation are not possible.[1] If there were some estimated maximum variability in the stress to failure for the parts, survivability might be obtained by using the highest survived stress as an overtest[2] level. However, such estimated maximum variabilities may also be unavailable. (If they were available, they could have obviated the excessively large step-stress sizes.) A common way to treat the data is to assume that one part failed just above the highest survival level and that all the other parts failed just below the next higher test level[3]. While this has the virtues of ease and simplicity, it leads to highly under-conservative part survivability estimates. The present article gives a more rigorous and conservative solution and presents it in the form of curves for easy application. Basically our method quantifies the common sense reasoning that if all the parts fail between two consecutive test levels, the probability distribution of stress to failure most likely has a mean between those levels and a s.d. which could not be much larger than the spacing between those levels (bin width).

2. MATHEMATICAL SOLUTION

2.1. Definition of Confidence.

Suppose the required survival probability at the specification stress is \( P_s \). We will show that if the survival probability at the specification stress is less than \( P_s \), the probability of our experimental result will be less than the quantity \( G^N \) where \( N \) is the sample size and \( G \) is a maximum possible probability that a single part will fail between two given stress levels. Adopting definitions consistent with the Lot Tolerance Percent Defective (LTPD) lot acceptance tests[4] the confidence is the probability of rejecting a lot whose parts have less than probability \( P_s \) of surviving the specification stress. Thus the confidence becomes:

\[
C = 1 - G^N
\]  

(1)

2.2. Model.

Let \( L_s \) be the specification level, \( L_l \) be the lower bound of the bin (the highest level at which all the parts survive) and let \( L_u \) be the upper bound of the bin (the lowest level at which all the parts fail). We will assume (as is implicitly done in the previous treatment of the problem) that the stress-to-failure follows normal or lognormal statistics. (If we are dealing with lognormal statistics the levels \( L_s, L_l \) and \( L_u \) refer to the logarithms of stress.) The relationship of the levels must be:

\[
L_s < L_1 < L_2
\]  

(2)

2.3. Approach.

If the survival probability at level \( L_s \) is less than \( P_s \), there is a s.d., \( \sigma \) which maximizes, \( G \), the probability that a single part will fail between levels \( L_s \) and \( L_u \). We know \( \sigma \) is finite because:
a) If \( U \) were infinite, the bin width in terms of \( U \) would be negligible giving a negligible probability of parts failing between levels \( L_1 \) and \( L_2 \). Thus we would contradict the condition that \( G \) be a maximum.

b) If \( U \) were infinitesimal, it would mean that all parts fail at essentially a single level between \( L_1 \) and \( L_2 \). This would imply that all parts survive the lower bound \( L_1 \) and, therefore, they must all survive the still lower level \( L_s \). Thus the survival probability at the specification stress would be 100\% and would contradict the condition that it be less than \( P_s \).

2.4. Solution.

We must (a) express the probability \( G \), in terms of \( L_s \), \( L_1 \), \( L_2 \), \( P_s \), and \( \sigma \), (b) maximize the expression with respect to \( \sigma \) and (c) show that the this value \( G \) is a maximum for all \( P < P_s \).

In the actual solution \( G \) is maximized with respect to a quantity related to \( \sigma \) through the relation \( r=(L_1-L_s)/\sigma \).

2.4.1. Analysis. The probability of a part failing between levels \( L_1 \) and \( L_2 \) is

\[
G = F\left(\frac{L_2-\mu}{\sigma}\right) - F\left(\frac{L_1-\mu}{\sigma}\right) \tag{3}
\]

\[
F\left( X \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} \exp\left(-\frac{x^2}{2}\right) \, dx
\]

\((i.e.) F \) is the cumulative normal distribution.) We may rewrite this equation as:

\[
G = F\left( Rr - \bar{F}_s \right) - F\left( r - \bar{F}_s \right),
\]

\[
where \quad R = \frac{L_2 - L_s}{L_1 - L_s} \quad \tag{4}
\]

\[
r = \frac{L_1 - L_s}{\sigma}
\]

\[
and \quad \bar{F}_s = \frac{\mu - L_s}{\sigma}
\]

Thus \( \bar{F}_s \) is such that \( F(\bar{F}_s)=P_s \).\( (i.e. \) if \( \bar{F}_s=1.282 \) then \( P_s = 0.9 \).)

Because \( L_s \geq L_1 \), we have \( R > 1 \) and \( G > 0 \). Furthermore, in any practical situation we would have \( r > 0 \) since the test level \( L_1 \) would be larger than the specification level \( L_s \). Taking the derivative of \( G \) with respect to \( r \):

\[
\frac{dG}{dr} = 0 = \frac{R}{\sqrt{2\pi}} \exp\left(-\frac{Rr - \bar{F}_s}{2}\right) - \left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(r - \bar{F}_s)^2}{2}\right)\right) \tag{5}
\]

In the domain \( R > 1 \) and \( r > 0 \), the solution of Equation (5) gives a unique worst case value, \( r_w \), for \( r \) and a unique maximum value for \( G \), \( G_{\text{max}} \):

\[
r_w = \frac{\bar{F}_s + \theta}{R + 1} \quad \text{and} \quad \frac{G_{\text{max}}}{G_{\text{max}}} = F\left( \frac{R\theta - \bar{F}_s}{R + 1} \right)
\]

\[
= F\left( \frac{\theta - R\bar{F}_s}{R + 1} \right) \tag{6}
\]

\(\text{where} \quad \theta = \sqrt{\bar{F}_s^2 + 2 \ln[R] \frac{R+1}{R-1}}\)

Figure 1 below shows a plot of \( P_s \) versus the maximum \( G \) for various values of \( R \).

To complete our derivation, we must show that \( G_{\text{max}} \) is a maximum for \( P_s \) and every probability less than \( P_s \). From either Figure 1 or an examination of Equation (6), we can see that \( G \) (and, therefore, \( G^{\theta} \)) decreases monotonically with \( P_s \).

2.5. Summary of Procedure.

If all \( N \) tested parts fail between levels \( L_1 \) and \( L_2 \) when the specification stress is \( L_s \) where \( L_s < L_1 < L_2 \), the confidence is computed as follows:

a) Compute \( R = (L_2-L_s)/(L_1-L_s) \).

b) Find \( G \) for the required survival probability, \( P_s \), from Figure 1.

c) Determine the confidence as \( C = 1 - G \).
2.5.1. **Sample Calculation.** Suppose the stress to failure is lognormally distributed and all we know about 10 tested parts is that all survive at 2 krad but fail at 8 krad. We wish the confidence for 99% survival at 1 krad. The logarithmic levels in decibels are 0, 3 and 9 dB(krad) for \( L_a, L_1 \) and \( L_2 \) respectively. Since \( 9/3 = 3 \), for 99% survival probability we take the curve for \( R=3 \) to get \( G=0.79 \). From Equation (1) the confidence is 90%. By contrast, the method of Reference 3, would give an overly-optimistic confidence of 97.5%. The difference between 90% and 97.5% can often have severe consequences.

Parameters \( \sigma \) and \( \mu \) are determined as follows: After \( r_a \) is determined from Equation (6), by algebraic manipulation of Equation (4) we get,

\[
\sigma = \frac{(L_1 - L_a)}{\sqrt{w_c}} \tag{7}
\]

\[
\text{and} \quad \mu = L_a + \frac{P_a \sigma}{\sqrt{w_c}}
\]

Note that the mean and s.d. accord with intuition. If computed, the s.d. would be \( \sigma = 2.19 \) dB(krad) which is somewhat less than \( 3 \) dB(krad) (the half-width of the bin). The mean would be \( \mu = 5.13 \) dB(krad) which places it between 3 and 9 dB(krad) (the boundaries of the bin).

3. **DISCUSSION**

3.1. **Reasonability of Answers.**

As we have seen, this calculation gives reasonable values of means and s.d.s. We will further show that any reasonable solution must encompass LTPD tests and overtests.

3.2. **LTPD Tests.**

If \( L_a \) is infinite (i.e. there is no upper test level), the ratio \( R \) becomes infinite, the parameter \( r_a \) approaches zero and therefore \( \sigma \) becomes infinite. Our test becomes an LTPD test because it is now a single level test where in units of the s.d. the difference between the test level and the specification level is negligible. Figure 1 shows that for infinite \( R \), the probability \( G \) becomes \( P \), and our confidence indeed becomes that of an N/O LTPD test. The method of Reference 3 does not, in the limit, become an LTPD test.

Interestingly, when \( R \) is finite, this test method gives higher confidence than LTPD tests. It may seem paradoxical that knowing an upper bound where all the parts fail increases the confidence of part survivability. However, part survivability is a function of both average behavior and variability. The upper bound provides an estimate of part variability. When used as a lot acceptance test, this method rejects lots on the
basis of both an unacceptable average performance and an unacceptable variability.

3.3. Overtesting.

Since this analysis derives a most pessimistic estimate for \( r = (L_1 - L_2)/\sigma \), any information showing that \( r \) must be either larger or smaller than \( r_{\text{crit}} \) would improve our confidence. The most practical example of such a situation is when engineering judgment places an upper bound on \( \sigma \) and, therefore, a minimum bound on \( r \) of \( r_{\text{min}} = (L_1 - L_2)/\sigma_{\text{max}} \). If \( r_{\text{min}} > r_{\text{crit}} \), we get:

\[
G_{\text{max}} = F \left( \frac{R r_{\text{min}} - \mu}{\sigma} \right) - F \left( \frac{r_{\text{min}} - \mu}{\sigma} \right) \tag{8}
\]

Equation (8) is true because a unique maximum for \( G \) means that the second derivative of \( G \) is negative for all physical values of \( R \) and \( r \). Thus, if \( r_{\text{min}} > r_{\text{crit}} \), then \( r_{\text{min}} \) gives the most pessimistic (i.e., largest) probability \( G \).

When there is no upper test level, we would have an overtest where \( G_{\text{max}} \), the probability that parts fail within our bin would now become \( P_r \), the survival probability at the overtest level \( L_1 \). Because \( L_1 \) is infinite, the ratio \( R \) becomes infinite and Equation (8) becomes a transformation of the overtest formula of Reference 2. Specifically if \( L_1 \) and \( L_2 \) are the respective logarithms of the test level and specification level (lognormal distribution):

\[
P_r = 1 - F \left( \frac{r_{\text{min}} - \mu}{\sigma} \right) = F \left( \frac{\mu - r_{\text{min}}}{\sigma} \right) \tag{9}
\]

Thus Equation (8) represents a new test using two overtest levels. When \( R \) is finite, Equation (8) will always give higher confidences than the overtest formula of Equation (9).

One might suppose that if we can place a minimum bound on \( \sigma \) such that \( r_{\text{min}} = (L_1 - L_2)/\sigma_{\text{min}} < r_{\text{crit}} \) we might likewise get a less pessimistic estimate of \( G \). However, this is not practical.

If we estimated \( \sigma_{\text{min}} \) to be much less than half the bin width we would have a conflict between experiment and engineering judgment.

3.4. Maximum Likelihood Methods.

If engineering judgment dictates some probability distribution for the parameter \( r \) rather than definite maxima or minima, it is possible to replace \( G_{\text{max}} \) with:

\[
G^n - \int G^n \Psi (r, X) \, dr \tag{10}
\]

where \( \Psi \) is a probability distribution in \( r \) characterized by parameters \( X \). The parameters \( X \) may be known in advance or estimated by maximum likelihood techniques. Such techniques are sometimes difficult to implement and are only mentioned here for completeness.

3.5. Further Ramifications.

If any parts survive the highest test level the step-stress analysis of Reference 1 would again be inapplicable but we propose a new technique similar to that of this method. We can only present the procedure here in broad outline. A lot acceptance test would be done with two overtest levels and attributes indicated in Table 1.

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Failure No.</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>Level</td>
<td>Devices per Device</td>
</tr>
<tr>
<td>0</td>
<td>Less than ( L_1 )</td>
<td>( N_0 )</td>
</tr>
<tr>
<td>1</td>
<td>Between ( L_1 ) and ( L_2 )</td>
<td>( N_1 )</td>
</tr>
<tr>
<td>2</td>
<td>More than ( L_2 )</td>
<td>( N_2 )</td>
</tr>
</tbody>
</table>

Lots would be rejected if \( N_r > 0 \) and we impose acceptance numbers, \( N_{rA} \) and \( N_{rB} \) such that \( N_{rA} \leq N_r \leq N_{rB} \). The probability of the experiment, \( G(\text{EXP}) \), would be expressed in terms of a) total sample size, b) all acceptance numbers, c) all test levels, d) the specification level and e) the survival probability at the specification level. We then maximize \( G(\text{EXP}) \) with respect to the s.d. applying the constraint that at the specification level the survival probability must be less than the required value, \( P_r \). Thus, the confidence of rejecting lots with unacceptable survival probabilities would be at least \( C = 1 - G(\text{EXP})_{\text{max}} \). This paper presents the solution for the easiest and most useful case where acceptance numbers on \( N_r \) and \( N_L \) are both zero.

3.6. Assumption of Normal And Lognormal Distributions.

The calculation should be insensitive to deviations from normal or lognormal probability distributions if the assumption
is approximately correct between the specification stress and the lowest stress at which all parts fail. Previous treatments of this problem are as least as sensitive as the present approach to errors in assuming normal or lognormal probability distributions.

3.7. Inhomogeneous Inspection Lots.

The inspection lot may include several production lots and display a multi-modal distribution of stress to failure. In cases where precise bounds for stress to failure on each part have been determined, probability plots can expose an inhomogeneous inspection lot. However, when all the parts fail within a single bin, such plots are not possible. Since the present method bases the survivability on a most pessimistic estimate of the s.d., we will still generally get conservative probabilities and confidences. However, the topic of inhomogeneous inspection lots and outliers concerns the previous treatment of this problem as well as any treatment which is not distribution free.

4. CONCLUSIONS

We have presented a rigorous approach to a common problem in parts testing and have arrived at a solution which is both more precise and more conservative than previous solutions. Our method is satisfying because it is a general approach that includes both overtesting and LTPD testing. It further suggests a still more general type of overtest which should give better results than present methods and which would be useful even when there are no reliable estimates of maximum standard deviations.

5. REFERENCES


