A Conceptual Model of Single-Event Gate-Rupture in Power MOSFET's

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Abstract

A physical model of hole-collection following a heavy-ion strike is proposed to explain the development of oxide fields sufficient to cause single-event gate rupture in power MOSFET's. It is found that the size of the maximum field and the time at which it is attained are strongly affected by the hole mobility.

1. Introduction

For the first time, a simple model of hole build-up and its induced image charge in the gate electrode of a power MOSFET is shown to lead to large enough oxide fields to cause oxide breakdown. These holes are collected from the plasma sheath of electron-hole pairs generated along the strike path of an incident heavy ion. This model replaces the picturesque but unquantifiable visualization of the sheath of mobile pairs as a "plasma wire" or a "depletion-layer collapse" that short-circuits the depletion-layer voltage, allowing the drain bias to become applied directly across the oxide.

In power MOSFET's, two types of single-event damage are known: single-event burnout (SEB) and single-event gate rupture (SEGR). In the case of SEB, the currents stemming from charge collection cause a voltage drop sufficient to turn "on" a parasitic bipolar transistor inside the device, a parasitic inherent in the construction of the power MOSFET. If the strike occurs while the device is subject to a large drain bias, sufficient carrier multiplication occurs to cause runaway, creating a short-circuit through the MOSFET that allows current from the external power supply to destroy the device [1].

In the case of SEGR, the ion-generated electron-hole pairs are separated by the applied bias. For discussion, assume an n-channel device with gate grounded and drain positively biased, as shown in Fig. 1.

![Figure 1: Power MOSFET structure, showing an ion-strike filament at the center of the n-neck region, with holes moving upward and electrons downward under the influence of the positive drain voltage.](image)

The electrons are drawn toward the drain and the holes are driven toward the gate, along the axis of the strike filament. The effects of this charge collection commonly are visualized by picturing the sheath of electron-hole pairs surrounding the ion track as a conducting filament capable of short-circuiting the drain to the Si-SiO₂ interface. The

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idea of the strike filament as a "depletion-layer collapse" [2] or as a "plasma wire" [3,4] grew out of modeling of single-event upsets and was applied to SEGR by Hohl and Galloway[5]. As a result of this localized short-circuiting action of the filament, the ion strike results in a large fraction of the drain voltage dropping across the gate oxide, causing an increase in oxide field in the immediate vicinity of the filament.

If the transient increase in oxide field is large enough and lasts long enough, oxide breakdown occurs near the filament. A significant fraction of the charge stored in the MOSFET capacitor then can discharge, elevating the temperature in the neighborhood of the filament, locally destroying the oxide, short-circuiting the gate to the substrate.

\[
E_{CR} = \frac{41}{(\text{LET})^{1/2}} \times 10^6 \text{V/cm} 
\]

where LET is the linear energy transfer of the ion in MeV-cm²/mg. A typical test ion is 285 MeV Br, which has an LET of 37 MeV-cm²/mg, resulting in an \( E_{CR} \) of 6.7 MV/cm. This figure can be compared to a typical intrinsic breakdown strength of 8 – 12 MV/cm² for gate oxides grown on surfaces of typical microroughness [8]. Fischer[6] tested (1) on p-channel power MOSFETs with source shorted to drain and positive gate bias. In this configuration, the MOSFET behaves like an inverted MOS capacitor. Wrobel[7] observed (1) in studies of gate rupture in both accumulated and inverted MOS capacitors.

The data of Fig. 2 show that the critical field for SEGR in power MOSFET’s decreases with an increase in drain-to-source bias \( V_{DS} \), even when the device is biased in inversion. However, these devices with doping levels \( \approx 10^{19} \text{cm}^{-3} \) remain in inversion despite the \( V_{DS} \)-variation, so \( V_{DS} \) does not affect the oxide field prior to the ion strike. That is, Fig. 2 shows that when drain-to-source bias is applied, SEGR occurs at lower pre-strike critical oxide fields.

As mentioned earlier, the "plasma wire" model of the strike filament provides a picturesque view of how an oxide field approaching \( E_{CR} \) could arise in a localized region during or after the strike by short-circuiting the drain voltage to the Si-SiO₂ interface in the vicinity of the oxide-end of the filament. This model is misinterpreted, however, if the plasma filament is thought of simply as a "wire". The mechanism that transfers the applied voltage from the substrate to the oxide is the piling up of holes at the Si-SiO₂ interface. Analysis of this hole storage leads to the simple model discussed below.

3. Conceptual Model

Following an ion strike in the n⁺ neck region, the electrons are drawn toward the drain by the positive applied bias, and the holes are driven to the oxide interface beneath the grounded gate electrode (see Fig. 1). The electrons experience a spreading resistance as they drift and diffuse from the localized filament into the entire drain region. The resulting localized resistive (I-R) voltage drop accompanying the electron flow pushes the equipotentials deeper. That is, at the drain-end of the filament, the drain voltage is spread out over a longer distance, leading to an overall lowering of the field [2,3].
To understand the increase in oxide field, we consider hole collection. Holes are driven toward the oxide-end of the filament, at the interface between the n-neck region and the gate oxide, where they induce an image charge in the gate oxide, increasing the oxide field. This "pool" of holes at the interface is fed by the continued hole collection from the strike filament, and spreads radially across the interface with time, moving toward the p-body region, which is at ground potential.

A simple circuit model of the situation is shown in Fig. 3. The lumped capacitor Cgs in this circuit represents the storage capability of the interfacial region neighboring the oxide-end of the filament. The resistor Rs represents the leakage path along the Si-SiO2 interface from the strike filament to the grounded body. The extent of the oxide field build-up depends on the difference between the arrival rate of the holes (determined by \( I_1(t) \)) and the exit rate of the holes to the ground contact, determined by the RC-time constant of the circuit, \( \frac{1}{R_s} \).

The basic model assumes that the strike filament forces a current \( I_1(t) \) to the surface of the MOSFET. Although a general time dependence can be used, for simplicity \( I_1(t) \) is modeled as a simple expression:

\[
I_1(t) = I_{1n} + \left( I_{10} - I_{1n} \right) e^{-t/T} \tag{2}
\]

where \( T \) is the filament lifetime, determined by drift and diffusion within the plasma sheath, \( I_{1n} \) is the filament current at \( t=0 \), and \( I_{10} \) is the filament current following the early transient decay (\( t>T \)). During the time when oxide field is large, this form for \( I_1(t) \) fits the results obtained from numerical simulations using MEDICI (the TMA version of PISCES), as discussed later. Of course, for long times the current must vanish, unlike (2). Because the RC-time constant of this circuit is long compared to the filament lifetime \( T \) (typically a picosecond or so), the voltage across the capacitor will rise far enough to allow oxide fields in excess of the intrinsic breakdown strength.

The ion strikes most likely to cause SEGR occur far enough from the p-body that a considerable "pool" of holes collects before diffusion brings them into contact with the p-body. Under these conditions, the interfacial storage capacitor is connected to ground via a distributed RC-line, representing the surface inversion layer. This distributed RC-line models diffusion from the filament toward ground, rather than allowing immediate access to ground, as implied by the lumped circuit of Fig. 3. The next section describes the modeling of this distributed RC-line by a charge-sheet model developed to describe the hole collection from the filament by the body contact. The charge-sheet model shows that the hole collection satisfies a nonlinear diffusion equation. The nonlinearity increases the diffusion of the holes at large hole densities due to their self-field.

4. A Charge-Sheet Model for Hole Collection

At the Si-SiO2-end of the filament, we assume that the current flows radially outward from the filament toward ground in an interfacial charge sheet. A distributed RC-circuit corresponding to this analysis is shown in Fig. 4.

![Figure 4. A distributed RC-circuit model that generalizes the lumped circuit of Fig. 3. The parameters of the distributed RC-line are calculated using a charge-sheet model that relates the distributed resistance to the hole density/cm2 in the inversion layer.](image)

The distributed part of this circuit is modeled using a charge-sheet description of the inversion layer, analogous to that used for the MOSFET [9]. This approach leads to a nonlinear diffusion equation for the transport of the holes radially away from the strike filament, along the interface, toward the grounded body region. The mathematical description of this model is contained in the Appendix.

5. Numerical Simulations

Using MEDICI, numerical simulations of an ion strike were made in a cylindrical (r,z) geometry. The ion is taken to be normally incident, with the centerline of its track coincident with the z-axis. The ion track at \( t=0 \) is modeled by placing a filament of electron-hole pairs along the z-axis from the gate to the drain in the region \( 0\leq r < r_f(t=0) \), where \( r_f(t=0) = 0.124 \mu m \) in our simulation. For times \( t>0 \), the mobile carriers are transported in the self-consistent fields according to the usual drift + diffusion equations. This treatment of the carriers is not valid for very short times, as it ignores the transit time of the ion and the complicated processes that allow the initially energetic carriers to thermalize. However, the oxide field peaks in a time of the order of picoseconds, when most of these transients will be over. An idealization of a power MOSFET is used with a cylindrical geometry. The source contact of the power MOSFET is irrelevant, because it is shorted to the body contact, and because the holes are collected by the body, which is grounded. The body is modeled as a cylindrical interfacial p-contact of radius \( r_ea \), with \( a = 14.64 \mu m \) in our simulation. Electrons are collected by the drain, which is positively biased. The drain bias is sufficient to completely deplete the device, and before the ion strike all the equipotentials are parallel to the interface. A diagram of
our geometry is shown in Fig. 5, for an arbitrary choice of dimensions.

![Cylindrical geometry](image)

Figure 5. Cylindrical geometry for numerical simulations. The body contact is idealized as a shallow, cylindrical ring around the axis of the strike filament. To simplify analysis, the device is chosen to be fully depleted prior to the strike. The drain bias is 150V, gate and body are grounded, and the doping level is $5 \times 10^{24}/\text{cm}^3$.

The initial track density was chosen as $1.2 \times 10^{19}/\text{cm}^3$, which is not large enough for 285 MeV Br. (For this ion, the electron-hole pair density should be $2.4 \times 10^{10}/\text{cm}$ or $5 \times 10^{19}/\text{cm}^3$ for the chosen radius of 0.124μm.) However, larger densities caused non-convergence of the solutions.

![Hole density vs. radius](image)

Figure 6. Hole density at the interface in units of $10^{19}/\text{cm}^3$ vs. radial distance with time as parameter.

In Fig. 6 the calculated hole density/cm$^3$ at the interface is plotted vs. distance for several times following the ion strike. As Fig. 6 shows, The holes diffuse radially toward the body, but they do not reach ground during the time the oxide field is large. Unlike the lumped circuit of Fig. 3, the distributed circuit of Fig. 4 includes this diffusion.

The interfacial hole diffusion is accompanied by subsurface radial expansion of the filament. Isoconcentration lines for the holes at 10 ps are illustrated in Fig. 7.

![Hole concentration contour](image)

Figure 7. Contours of constant hole concentration at a time of 10 ps. Comparison with Fig. 6 is not possible, because the high-density hole contours are compressed in a thin inversion layer, not visible in this figure. In the n-epitaxial layer, the holes move radially outward with time, at the same time that they are traveling toward the top of the filament. Insert: same contours at 1ps.

![Hole concentration contour](image)

Figure 8. The radius of the hole concentration contour $r(t)$ for \( p = N_0 = 5 \times 10^{24}/\text{cm}^3 \) vs. time at a depth of \( z = 10 \mu \text{m} \). Solid line A: one-dimensional, ambipolar, radial diffusion model [5]. Open circles B: MEDICI output. Dashed line C: radius \( r_C(t) \) of the interfacial storage region as determined from MEDICI by fitting (A3) of the Appendix. The curve C is for field-dependent mobility, the curve D for constant mobility.
As the oxide field is building up sufficiently to cause SEGR, the radial motion of the carriers 10μm below the interface is approximately described by the cylindrical ambipolar diffusion model of Hohl and Galloway [5]. In Fig. 8 a comparison of this diffusion model with the numerical results is shown.

The current flowing from the filament to the interfacial storage capacitor is found by integrating the hole charge over the entire filament, and differentiating it to obtain the current $I_f(t)$. The computed charge at various depths is shown in Fig. 9.

It can be seen that the charge at deeper z-values decays more rapidly, because holes are driven toward the surface, depleting the deeper part of the filament first. Integrating over depth, the total charge in the filament as a function of time is shown in Fig. 10. Taking the derivative of the hole charge, $Q(t)$, we find $I_f(t)$ is fitted well by (2). For the particular example we are presenting here, we find $I_{fb} = 6.3 mA$, $I_{on} = I_{on}$, and the filament time constant is not required.

The collecting pool of holes at the oxide-end of the strike filament induces an image charge in the gate electrode. The resulting field between the holes and their image charge appears in the oxide, raising the oxide field to values approaching the intrinsic breakdown field, as shown in Fig. 11.

It is apparent that the field is much lower and peaks much earlier in time, showing that the saturation of the hole mobility at large lateral fields is important.

Figure 12 shows the same results for the case of constant mobility (no velocity saturation). It is apparent that the field is much lower and peaks much earlier in time, showing that the saturation of the hole mobility at large lateral fields is important.

6. Comparison with Simple Model

The distributed circuit model described in the Appendix has been solved using Newton’s method, as outlined in the Appendix. The results agree qualitatively with the numerical results of MEDICI. For example, the result for the oxide field vs. time from the simple model is shown in Fig. 13. Although the agreement is rough, the two calculations provide the same order of magnitude for the field, the time scales, and the rate of diffusion of the holes from the filament toward the ground contact. It is hoped that the
agreement can be improved by a more careful treatment of the interfacial storage capacitor, and of the radial expansion of the filament with time.

![Figure 13. Oxide field vs. radial position with time as parameter from the distributed circuit model with μ = 500cm²/Vs.](image)

7. Conclusions

A numerical simulation of an ion strike in a power MOSFET has been interpreted in terms of a simple model of hole collection. It has been shown that oxide fields larger than the intrinsic breakdown strength of the oxide can arise from the holes collecting at the interface and their image charge in the gate electrode. These high fields persist for times of the order of picoseconds. It is not known how long these fields must persist to initiate SEGR.

The model shows that the extent of the field build-up depends on several factors which must be considered if SEGR sensitivity is to be reduced. SEGR sensitivity will be reduced if the diffusion of the holes to ground is easy, allowing the holes to drain off before too many are collected at the interface. This point is well illustrated by the mobility dependence of the field build-up, as discussed earlier. Removal of holes could be expedited to make SEGR less likely if the hole mobility at the interface were increased, or if the geometry of the neck region were altered to make ground contacts closer, or if impediments in the path to ground were avoided, such as n⁺-implants from other parts of the structure. SEGR sensitivity also can be reduced if the oxide or depletion-layer capacitances are reduced, not primarily because voltages are dropped across larger distances (e.g., larger oxide thicknesses), but because the circuit RC-time-constant is reduced, so fewer holes are stored at the surface. Additional reduction in SEGR sensitivity is provided if fewer of the generated holes reach the interface, e.g., if the recombination rate in the silicon substrate can be increased, or if subsurface hole collection can be provided.

8. Acknowledgements

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9. Appendix: Simple Charge-Sheet Model of SEGR in Power MOSFET

A1. Charge-Sheet Transport

In the oxide there is no charge so the potential \( \phi(r,z,t) \) satisfies Laplace's equation:

\[
\nabla^2 \phi(r,z,t) = 0
\]

where \( \phi(r,z,t) \geq 0 \) for the condition of large positive drain bias. The boundary conditions on \( \phi(r,z,t) \) at all times are

(i) At the gate electrode, \( \phi(r,z=d_{OX},t)=0 \). Here \( d_{OX} \) = oxide thickness.

(ii) At the Si–SiO₂–interface,

\[
-\left. \frac{\partial \phi(r,z,t)}{\partial z} \right|_{z=\infty} = \frac{q}{\varepsilon_{OX}} P(r,t)
\]

where \( P(r,t) \) is the hole density / cm² at the interface.

For \( r \leq r_f \), we assume that the hole distribution at the interface is described by

\[
qP(r,t)=\frac{Q_{BS}(t)}{\pi r_f(t)^2} \exp\left[-(r/r_f)^{1.8}\right] \frac{0.614946}{0.164946}
\]

where \( P(r,t) \) is the hole density / cm² at the interface.

The value of \( Q_{BS}(t) \) is determined by current balance as

\[
Q_{BS}(t)=\int_0^t \left[I_s(t')-2\pi r_f(t') J_r(t,t')\right] dt'.
\]

In this equation both \( I_s(t) \) and \( r_f(t) \) are given functions of time.

We must find \( J_r(t,t) \) by solving the transport equation that decides how fast the charge leaks off to ground. This equation is the continuity equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left(r J_r(r,t)\right) + \frac{\partial qP(r,t)}{\partial t} = 0
\]

In this equation the radial current density is taken from the charge-sheet model as
\[ J_r(r,t) = -D \frac{\partial}{\partial t} \left( \frac{qP(r,t)}{kT} \right) + \frac{\partial}{\partial r} \phi(r,z=0,t) + \frac{\partial}{\partial r} P(r,t) \]  

(A6)

with \( D \) = zero-field diffusion coefficient and with

\[ f(F_r) = \frac{1}{1 + (F_r/F_0)} \]

to account for the radial field (\( F_r = -\partial \phi / \partial r \)) dependence of the hole mobility. The continuity equation then becomes, using (A6) and (A2),

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r f(E_r) \left( \frac{q}{kT} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} \right) \right] = \frac{1}{D} \frac{\partial^2 \phi}{\partial t \partial z} \]  

(A7)

For large enough values of \( r \), we suppose that the gradual-channel approximation is valid, so \( \partial \phi / \partial z \approx -\phi / d_{ox} \). Then the continuity equation becomes

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r f(E_r) \left( \frac{q}{kT} \phi + 1 \right) \frac{\partial \phi}{\partial r} \right] = \frac{1}{D} \frac{\partial \phi}{\partial r} \]  

(A8)

Assuming the gradual-channel approximation works at \( r = r_f \), we find

\[ \frac{q}{\varepsilon_{ox}} \frac{P(r_f,t)}{\varepsilon_{ox} \pi r_f^2(t)} = \frac{Q_{ss}(t)}{\varepsilon_{ox} \pi r_f^2(t)} \cdot \frac{e^{-1}}{0.614946} \]

\[ = -\frac{\partial \phi(r_f,z,t)}{\partial z} \bigg|_{z=0} \approx \frac{\phi(r_f,z=0,t)}{d_{ox}} \]  

(A9)

which determines the end condition for the continuity equation at \( r = r_f \), in terms of \( Q_{ss}(t) \) and \( r_f(t) \):

\[ \phi(r_f(t),z=0,t) = 0.598 \cdot d_{ox} \frac{Q_{ss}(t)}{\varepsilon_{ox} \pi r_f^2(t)} = 0.598 \cdot \frac{Q_{ss}(t)}{C_{ss} \frac{r_f(t)}{kT/q}} \]

(A10)

with \( C_{ss} \left[ r_f(t) \right] = \varepsilon_{ox} \pi r_f^2(t) / d_{ox} = \) interfacial storage capacitance.

**A2. Solution Procedure**

To solve the equations one approach is as follows:

(i) Assume that \( J_r(r_f,t) = 0 \).

(ii) Find \( Q_{ss}(t) \) for all time steps \( t \) by integration of (A4) using the midpoint rule.

(iii) Set up the end condition using (A10) for \( \phi(r_f(t),z=0,t) \) given \( Q_{ss}(t) \) and \( r_f(t) \).

(iv) Solve (A7) using this end condition.

(v) From the solution of (A7), find \( J_r(r_f(t),t) \) for all time steps using (A6).

(vi) Recalculate \( Q_{ss}(t) \) from (A4) including \( J_r \) (i.e. return to step (ii)).

(vii) Repeat (ii) – (vi) until no change in \( Q_{ss} \) results.

To solve (A7) replace \( \phi \) by

\[ u(r,t) = \left[ 1 + q\phi / kT \right]^2 \]  

(A12)

We also introduce the normalized length \( x \) by

\[ x = r / a \]  

(A13)

where \( a = \) radius to the ground contact. We introduce the new time variable \( \tau \)

\[ \tau = D \tau / a^2 \]  

(A14)

so (A7) becomes

\[ \frac{\partial^2 u}{\partial x^2} + \left[ \frac{1 + \partial \ln f}{\partial x} \frac{1}{V_{4u}} \frac{\partial u}{\partial \tau} \right] = \frac{1}{\partial \tau} \]  

(A15)

where now \( f \) is given by

\[ f = \frac{1}{1 + \left( \frac{kT/q}{2a F_0} \right) \left( \frac{1}{V_{4u}} \frac{\partial u}{\partial \tau} \right)} \]  

(A16)

The initial condition for (A15) is

\[ u(x, \tau = 0) = 1 \]  

(A17)

and the end conditions are

\[ u(\text{xf}, \tau) = 1 \]  

(A18)

\[ u(\text{xf}, \tau) = \left[ 1 + \frac{2}{\pi} \frac{Q_{ss}(\tau)}{C_{ss}(\tau) kT/q} \right]^2 \]  

(A19)

with

\[ Q_{ss}(\tau) = \frac{a^2}{D} \int_0^r \left[ \frac{r_f(t) + \pi x_f(t)}{q d_{ox} D} \frac{\partial u}{\partial x} \bigg|_{x=x_f(t)} \right] \, dt \]  

(A20)

and

\[ C_{ss}(\tau) = \frac{\varepsilon_{ox}}{d_{ox}} \pi a^2 x_f^2(\tau) \]  

(A21)
A3. Numerical Solution by Newton's Method

We linearize (A15) by letting \( u = u_o + \delta u \) and \( f = f_o + \delta f \), where \( \delta u \) and \( \delta f \) are the small departures of the exact solution \( u \) from a guessed solution \( u_o \). Following linearization, (A15) becomes

\[
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial f}{\partial x} = & \frac{1}{x} \frac{\partial u}{\partial x} - \frac{1}{z} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial t} \\
= & \frac{1}{x} \left( \frac{\partial f}{\partial t} \right) - \frac{1}{x} \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u}{\partial t} \\
= & \frac{1}{x} \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial u}{\partial t} \right) \\
\end{aligned}
\]  

(A22)

We discretize (A22) neglecting \( \delta f \) as follows. Let \( \varepsilon \) = time step, \( \Delta = \text{space step}, s = \frac{\Delta x}{\varepsilon} \). Also, let \( u^{(i)}(x(i), t + \varepsilon) \) and let \( u^{(i+1)}(x(i+1), t + \varepsilon) \).

Then at time \( t + \varepsilon \),

\[
\begin{aligned}
u^{(i+1)(i+1)} & \left( 1 + \frac{\Delta}{2x(i)} + \frac{1}{4} \frac{f_x^{(i+1)}}{f_x^{(i+1)}} \right) \\
u^{(i+1)(i-1)} & \left( 1 - \frac{\Delta}{2x(i)} - \frac{1}{4} \frac{f_x^{(i+1)}}{f_x^{(i+1)}} \right) \\
u^{(i+1)} & \left( 2 + \frac{s}{2} \frac{f_x^{(i)}/u_x^{(i)}}{u_x^{(i)}} \right) \left[ 1 + \frac{u_x^{(i)}}{u_x^{(i)}} \right] \\
\end{aligned}
\]

(A23)

To estimate \( u_x^{(i)} \) we use

\[
u_x^{(i)} = \frac{1}{x} \left[ \frac{\partial u_x^{(i)}}{\partial x} \right] \varepsilon
\]

(A24)

Equation (A23) is to be solved iteratively at each time step as follows: we start with the guess \( u_x^{(0)}(i, t + \varepsilon) \equiv u(i, t) \).

Then we use this \( u_x^{(0)} \) in (A23) and solve to obtain \( u_x^{(i)}(i, t + \varepsilon) \). As the next guess we take \( u_x^{(i)}(i, t + \varepsilon) \) and \( u_x^{(i+1)}(i, t + \varepsilon) \). With this \( u_x \), again we solve (A23) for the second approximation \( u_x^{(i)}(i, t + \varepsilon) \). Then take \( u_x = u_x^{(i)} \) and so forth until the change in \( u(i, t + \varepsilon) \) is small. Once this occurs, we increment the time to \( t + \varepsilon \), find the new value of \( x(i, t + \varepsilon) \), redefine the index \( i \), and continue.

For the case \( i = 1 \), (A23) involves \( u(0) \). We find \( u(0) \) from (A19).

On the first pass through the loop of Section A3, we assume \( J_i = 0 \), so we obtain \( u(0) \) from (A19) with \( Q_{85} \) from (A20):

\[
Q_{85}(t + \varepsilon) = \int_0^{t+\varepsilon} \left[ f_x(\xi) \right] d\xi
\]

(A36)

with \( \xi = \text{a dummy variable of integration} \). On the next pass, in (A20) we need \( \frac{\partial u}{\partial x} \mid_{x=x_T} \).

10. References


