On the Regularity of Wavelets

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Abstract—The regularity index \( \alpha \) of the scaling functions \( \psi \), \( N = 2, 3, \cdots \) of multiresolution analysis introduced by Daubechies in 1988 is investigated. It is shown that \( 0.51 < \alpha_2 < 0.53 \) and \( \lim_{N \to \infty} \frac{\alpha_N}{N} = 1 - \log 3/(2 \log 2) \).

Index Terms—Wavelets, regularity index, dilation equation, Fourier transform, scaling functions.

I. INTRODUCTION

If \( f \) is a trigonometric polynomial with \( f(0) = 1 \) then the infinite product

\[
F(z) = \prod_{j=1}^\infty f(2^{-j}z)
\]

defines an entire function \( F \) (e.g., see [6, Lemma 2.3]). In the special case \( f(z) = \cos z \) we obtain

\[
\prod_{j=1}^\infty \cos \left( 2^{-j}z \right) = \frac{\sin z}{z}.
\]

This method is used in [2] to solve dilation equations of multiresolution analysis by functions with compact support. The solutions (called scaling functions) are Fourier transforms of infinite products of the type previous shown. Then wavelet bases of \( L^2(\mathbb{R}) \) can be defined in terms of these scaling functions. One is interested to find scaling functions with compact support as smooth as possible which then leads to the problem to investigate the behavior of \( F \) for \( z \to \pm \infty \). In this correspondence, we treat this problem for the special trigonometric polynomials \( f = f_N, N \in \mathbb{N} \), given in [2, Section 4]. These polynomials are products

\[
f_N(z) = \left( \frac{1}{2} \left( 1 + e^{iz} \right) \right)^N g_N(z),
\]

where \( g_N \) is a trigonometric polynomial satisfying \( g_N(0) = 1 \) and

\[
|g_N(z)|^2 = P_N \left( \sin^2 \left( \frac{1}{2} z \right) \right),
\]

where

\[
P_N(y) := \sum_{j=0}^{N-1} \left( N - 1 + j \right) y^j.
\]

Here and in the following the arguments of our functions are always real numbers. The function \( g_N \) can be made unique by additional conditions which, however, are not needed in this correspondence because we only use \( |g_N(z)| \). If we denote the functions of (1.1) corresponding to \( f_N, g_N \) by \( F_N \) and \( G_N \), respectively, then (1.2) shows that

\[
|F_N(z)| = \left| \sin \left( \frac{1}{2} z \right) \right|^N \left| G_N(z) \right|.
\]

As in [2, p. 981, (4.28)] (there is a misprint in that formula: the power \( 1 + \alpha \) has to be replaced by \( \alpha \)), let \( \alpha_N \) (the "regularity index" of the Fourier transform of \( F_N \)) be the supremum of all \( \beta \) such that

\[
\int_{-\infty}^{\infty} (1 + |t|)^\beta |F_N(t)| \, dt < \infty.
\]

Then the problem is to find upper and lower bounds of \( \alpha_N \). For instance, it is known that \( \alpha_2 \geq 0.5 \) (see [2, p. 984]). In Section III, we improve this to

\[
0.51 < \alpha_1 < 0.53.
\]

In a remark on p. 984 of [2] we find the value \( \alpha_2 = 2 - (\log (1 + \sqrt{3}))/(\log 2) = 0.55 \cdots \). According to [3] this value corresponds to a different definition of the regularity index \( \alpha_2 \). There \( \alpha_N \) is the supremum of all \( \alpha \) such that the Fourier transform of \( F_2 \) belongs to the space Lip-\( \alpha \) of functions \( f \) satisfying

\[
|f(x) - f(x + t)| \leq C(1 + |t|)\beta.
\]

In particular, one is interested in the asymptotic behavior of \( \alpha_N \) as \( N \to \infty \). It is known that \( \alpha_N/N \geq 0.1936 + O(N^{-1} \log N) \), see [2, p. 983]. However, the limit of \( \alpha_N/N \) given on p. 983 of [2] has turned out to be wrong [3]. In Section IV of this paper, we will prove that

\[
\lim_{n \to \infty} \frac{\alpha_N}{N} = 1 - \frac{\log 3}{2 \log 2}.
\]
This result does not depend on the particular choice of the regularity index, i.e., it remains the same if the sequence \( \alpha_N \) is replaced by some other sequence \( \tilde{\alpha}_N \) such that \( \alpha_N - \tilde{\alpha}_N \) is a bounded sequence. The value of the limit (1.7) was found independently by another method by A. Cohen and J. P. Conze. Some results of A. Cohen will appear in [1]. Since the estimations are delicate at various points it is certainly a good idea to have two different proofs of (1.7).

Our estimates of \( a_N \) are elementary. They are based on a simple proposition that is proved in Section II.


II. A BASIC PROPOSITION

The definition (1.3) of \( P_N \) shows that \( P_N(y) \geq 1 \) for \( 0 \leq y \leq 1 \). Hence we find that

\[ |F_N(z)| \geq \left| \frac{\sin \left( \frac{1}{2} z \right)}{\frac{1}{2} z} \right|^N, \]

which yields the rough estimate \( \alpha_N \leq N - 1 \). To find better bounds we will use the following result.

**Proposition 1:** Let \( N \) be a fixed positive integer, and let \( \lambda > 0 \).

a) If there is a positive constant \( C \) such that

\[ \int_0^{2\pi} |\sin (2^{m-1} t)|^N \prod_{k=0}^{m-1} |g_N(2^k t)| \, dt \geq C \lambda^m, \]

for all \( m \in \mathbb{N}, \) (2.1)

then

\[ \alpha_N \leq N - \frac{\log (2\lambda)}{\log 2}. \] (2.2)

b) If there is a positive constant \( C \) such that (2.1) holds with the inequality reversed then (2.2) holds with the inequality reversed.

**Proof:** a) Since \( N \) is fixed during the proof let us agree to suppress the subscript \( N \) of the various functions. Define

\[ \tilde{G}(z) := |\sin \left( \frac{1}{2} z \right)|^N |G(z)| \]

and \( H(z) := \int_0^z \tilde{G}(t) \, dt. \)

Then \( G(2t) = g(t) G(t) \) yields

\[ H(2^m 2\pi) = 2^m \int_0^{2\pi} \tilde{G}(2^m t) \, dt \]

\[ = 2^m \int_0^{2\pi} \left| \sin (2^{m-1} t) \right|^N \prod_{k=0}^{m-1} |g(2^k t)| \, dt. \] (2.3)

Now \( |G(t)| \geq 1 \) and assumption (2.1) shows that \( H(2^m 2\pi) \geq C(2\lambda)^m. \) Since \( H \) is increasing this implies that there is a positive constant \( C_1 \) such that

\[ H(z) \geq C_1 z^{\log (2\lambda)/\log 2}, \]

for \( z \geq 1. \) (2.4)

Integration by parts gives, for \( z \geq 1,

\[ 2^{-N} \int_1^z r^\beta |F(t)| \, dt = \int_1^z r^{\beta - N} \tilde{G}(t) \, dt \]

\[ = t^{\beta - N} H(t) \frac{1}{(N - \beta)} \int_1^t t^{-N+1} H(t) \, dt. \] (2.5)

Now (1.5) implies \( N - \beta > 0, \) and thus (2.4) and (2.5) give

\[ \beta - N + \log (2\lambda)/\log 2 \leq 0. \]

This proves (2.2).

b) Let \( C_2 := \max_{0 \leq t \leq \pi} |G(t)|. \) Then (2.3) shows that

\[ H(z) \leq C_2 z^{\log (2\lambda)/\log 2}, \]

for \( z \geq 1. \)

Now (2.5) and \( |F(z)| = |F(-z)| \) shows that this implies (2.3) with the inequality reversed.

**Proposition 1** is closely related to the estimates used in the Appendix of [2]. The lower bound for the regularity index that is found in [1] is an immediate consequence of the proposition: If \( B_m := \sup_{x \in \mathbb{R}} \prod_{k=0}^{m-1} |g(2^k x)|, \) \( b_m = \log B_m / \log 2 \) and \( b = \inf_m b_m, \) then the left-hand side of (2.1) is \( \leq CB_m \leq C(2^{b+1})^m. \) From Proposition 1 follows that the regularity index \( \alpha \) satisfies \( \alpha \geq N - 1 - b - \varepsilon (\varepsilon > 0), \) in agreement with Cohen’s result.

III. LOWER AND UPPER BOUNDS FOR \( \varepsilon/2 \)

We now apply Proposition 1 to find lower and upper bounds for the regularity index \( \alpha_2. \) Let us write \( g \) in place of \( g_2 \) in this section. Since we can take

\[ g(z) = p + q e^{iz}, \]

where

\[ p := \frac{1}{2} (1 + \sqrt{3}), \quad q := \frac{1}{2} (1 - \sqrt{3}), \] (3.1)

we obtain

\[ |g(z)| \geq \Re g(z) = p + q \cos z \]

\[ = p + \frac{q}{2} (e^{-iz} + e^{iz}) = h_i(z). \] (3.2)

Hence,

\[ \prod_{k=0}^{m-1} |g(2^k t)| \geq \prod_{k=0}^{m-1} h_i(2^k t) = \sum_{l=-2^{m-1}}^{2^{m-1}} a_l e^{il}, \] (3.3)
It is easy to see that \( a \in [\pm 1] \). We now use
\[
\sin^2 x = -\frac{1}{2}e^{-2ix} + \frac{1}{2}e^{2ix}
\]
to obtain
\[
\sin^2 \left( (m-1) \sum_{k=0}^{m-1} |h_k(2^k t)| \right) dt = \sum_{l} \tilde{a}_{lm} e^{i\pi l}
\]
where \( \tilde{a}_{om} = \frac{1}{2}p^m \). Hence,
\[
\int_{0}^{2\pi} \sin^2 \left( (m-1) \sum_{k=0}^{m-1} |g(2^k t)| \right) dt = \pi p^m.
\]
Now Proposition 1a) yields
\[
\alpha_2 \leq 2 - \frac{\log(1 + \sqrt{3})}{\log 2} = 0.5500 \ldots. \tag{3.4}
\]
By accident, this upper bound for \( \alpha_2 \) is the exact value for a differently defined regularity index, see [2, p. 984]. However, we will now show that the inequality (3.4) is strict.

First we sharpen the estimate (3.2). Write \( g(z) = x + iy, x, y, z \in \mathbb{R} \). Then (3.1) shows that \( x + iy \) lies on a circle centered at \( p \) with radius \( -q \). Hence,
\[
\sqrt{x^2 + y^2} - x = \frac{y^2}{\sqrt{x^2 + y^2} + x} \geq \frac{1}{2(p - q)} y^2. \tag{3.5}
\]
This yields
\[
|g(z)| \geq p + q \cos z + \frac{q^2}{2(p - q)} \sin^2 z
\]
\[
= a + b(e^{-iz} + e^{iz}) + c(e^{-iz} + e^{iz}) = : h_2(z)
\]
where
\[
a = p + \frac{q^2}{4(p - q)}, \quad b = \frac{1}{2}q, \quad c = -\frac{q^2}{8(p - q)}.
\]
Write
\[
\prod_{k=0}^{m-1} h_2(2^k z) = \sum_{l=-2^{m+1}+2}^{2^{m+1}+2} a_{lm} e^{ilz}.
\]
Then, by (3.6) and (3.3),
\[
\int_{0}^{2\pi} \sin^2 \left( (m-1) \sum_{k=0}^{m-1} |g(2^k t)| \right) dt \geq -\frac{1}{4} b_{-1m} + \frac{1}{2} b_{0m} - \frac{1}{4} b_{1m}, \tag{3.7}
\]
where \( b_{jm} = a_{p^m}, j = -1, 0, 1 \). To find \( b_{jm} \) we use that
\[
\begin{pmatrix}
  b_{-1m} \\
  b_{0m} \\
  b_{1m}
\end{pmatrix} = A^m \begin{pmatrix}
  0 \\
  1 \\
  0
\end{pmatrix},
\]
where
\[
A = \begin{pmatrix}
  b & c & 0 \\
  b & a & b \\
  0 & c & b
\end{pmatrix}.
\]
The eigenvalues of \( A \) are
\[
\lambda_{1,2} = \frac{1}{2} \left( a + b \pm \left( (a - b)^2 + 8bc \right)^{1/2} \right), \quad \lambda_3 = b,
\]
which yields
\[
\lambda_1 = 1.3876 \ldots, \lambda_2 = -0.1852 \ldots, \lambda_3 = -0.1830 \ldots.
\]
Now
\[
\begin{pmatrix}
  1 \\
  -1 \\
  0
\end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix}
  c \\
  -b \\
  c
\end{pmatrix} + \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix}
  c \\
  b \\
  -c
\end{pmatrix}
\]
and (3.8) gives
\[
\begin{pmatrix}
  b_{-1m} \\
  b_{0m} \\
  b_{1m}
\end{pmatrix} = \frac{\lambda_1^m}{\lambda_1 - \lambda_2} \begin{pmatrix}
  1 \\
  -b \\
  c
\end{pmatrix} + \frac{\lambda_2^m}{\lambda_2 - \lambda_1} \begin{pmatrix}
  1 \\
  b \\
  -c
\end{pmatrix}.
\]
We obtain
\[
-\frac{1}{4} b_{-1m} + \frac{1}{2} b_{0m} - \frac{1}{4} b_{1m} = \frac{1}{2} \lambda_1 - b - c \lambda_1^m
\]
\[
+ \frac{1}{2} \lambda_2 - b - c \lambda_2^m.
\]
Since the factor of \( \lambda_1^m \) is positive, (3.7) shows that we can take \( \lambda = \lambda_1 \) in Proposition 1a). Hence,
\[
\alpha_2 \leq 1 - \frac{\log \lambda_1}{\log 2} = 0.52739 \ldots. \tag{3.9}
\]
We now proceed to find lower bounds for \( \alpha_2 \). Using the Cauchy–Schwarz inequality we find
\[
\left( \int_{0}^{2\pi} \prod_{k=0}^{m-1} |g(2^k t)| dt \right)^2 \leq 2\pi \int_{0}^{2\pi} \prod_{k=0}^{m-1} |g(2^k t)|^2 dt = 4\pi^2 2^m.
\]
The latter equality follows from \( |g(z)|^2 = 2 - \frac{1}{2} (e^{iz} + e^{-iz}) \). Hence we can take \( \lambda = \sqrt{2} \) in Proposition 1b) which yields
\[
\alpha_2 \leq 0.5. \tag{3.10}
\]
This is the lower bound for \( \alpha_2 \) given in [2, p. 994].
We now improve this lower bound by using an inequality similar to (3.5). If \( g(z) = x + iy, x, y, z \in \mathbb{R} \) then

\[
\sqrt{x^2 + y^2} - x = \frac{y^2}{\sqrt{x^2 + y^2} + x} \leq \frac{1}{2(p + q)} y^2 = \frac{1}{2} y^2,
\]

which yields

\[
|g(z)| \leq h_3(z) := a + b(e^{iz} + e^{-iz}) + c(e^{2iz} + e^{-2iz})
\]

where

\[ a = p + \frac{1}{2} q^2, \quad b = \frac{1}{2} q, \quad c = -\frac{1}{4} q^2.\]

Using the matrix (3.8) again we find

\[
\int_0^{2\pi} \prod_{k=0}^{m-1} |g(2^k t)| \ dt \leq \int_0^{2\pi} \prod_{k=0}^{m-1} h_3(2^k t) \ dt \leq C M^m
\]

where \( \lambda \) is the spectral radius of \( A \). Then Proposition 1 b) yields

\[
\alpha_2 \geq 1 - \frac{\log \lambda}{\log 2} = 0.51109 \ldots.
\]

IV. THE ASYMPTOTIC BEHAVIOR OF THE SEQUENCE \( \alpha_N \)

We first derive a lower bound for \( \alpha_N \) that is based on the following two lemmas.

**Lemma 1:** Let \( p:[0,1] \rightarrow [0,1] \) be a function such that

\[
p(y) \leq \frac{1}{4} \quad \text{on} \quad 0 \leq y \leq \frac{3}{4}
\]

and

\[
p(y) p(4y(1-y)) \leq \frac{1}{4} \quad \text{on} \quad \frac{1}{2} \leq y \leq 1.
\]

Then,

\[
\prod_{j=0}^{m-1} p(\sin^2(2^j x)) \leq \left( \frac{3}{4} \right)^{m-1}, \quad \text{for all} \quad x \in \mathbb{R}, \quad m \in \mathbb{N}.
\]

**Proof:** Since \( \sin^2(2x) = 4 \sin^2 x(1 - \sin^2 x) \), this inequality can be rewritten as

\[
\prod_{j=0}^{m-1} p(\sin^2(2^j x)) \leq \left( \frac{3}{4} \right)^{m-1}, \quad \text{for} \quad 0 \leq y \leq 1. \tag{4.1}
\]

where \( q_j(y) = y \) and \( q_j^i \) is the jth iterate of \( q(y) = 4y(1-y) \).

We prove (4.1) by induction on \( m \). If \( m = 1 \), then (4.1) is true. Assume that (4.1) holds for all \( m \leq M \). If \( 0 \leq y \leq \frac{1}{2} \) and \( z := q_j(y) \) then

\[
\prod_{j=0}^{M} p(q_j^i(y)) = p(y) \prod_{j=1}^{M-1} p(q_j^i(y)) = p(y) \prod_{j=0}^{M} p(q_j^i(z)) \leq \frac{3}{4} \left( \frac{3}{4} \right)^{M-1} \cdot
\]

If \( \frac{1}{2} \leq y \leq 1 \) and \( z := q_j^i(y) \) then

\[
\prod_{j=0}^{M} p(q_j^i(y)) = p(y) \prod_{j=1}^{M-1} p(q_j^i(y)) = p(y) \prod_{j=0}^{M} p(q_j^i(z)) \leq \frac{3}{4} \left( \frac{3}{4} \right)^{M-1} \cdot
\]

Hence, (4.1) holds for \( m = M + 1 \), which completes the proof.

**Lemma 2:** The polynomial \( P_N \) of (1.3) satisfies the inequality

\[
P_N(y) \leq 4^{N-1}(\max \{ \frac{1}{2}, y \})^{N-1}, \quad 0 \leq y \leq 1.
\]

**Proof:** From

\[
y^N P_N(1-y) + (1-y)^N P_N(y) = 1,
\]

(see [2, Proposition 4.5]) it follows that \( P_N(\frac{1}{2}) = 2^{N-1} \). Since \( P_N \) is increasing on \([0, 1]\), we conclude that \( P_N(y) \leq 2^{N-1} \) for \( 0 \leq y \leq \frac{1}{2} \). If \( \frac{1}{2} \leq y \leq 1 \) then

\[
P_N(y) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j = \sum_{j=0}^{N-1} \binom{N-1+j}{j} 2^{-j}(2y)^j
\]

\[
\leq (2y)^N \sum_{j=0}^{N-1} \binom{N-1+j}{j} 2^{-j}
\]

\[
= (2y)^N P_N(y) \left( \frac{1}{2} \right) = (4y)^N - 1.\]

We apply Lemma 1 to

\[
p(y) := \max \{ \frac{1}{2}, y \}.
\]

A simple calculation shows that this function satisfies the assumptions of the lemma. Combining Lemmas 1 and 2 we obtain

\[
\prod_{j=0}^{m-1} P_N(\sin^2(2^j t)) \leq 4^{m(N-1)} \left( \prod_{j=0}^{m-1} p(\sin^2(2^j t)) \right)^{N-1}
\]

\[
\leq \left( \frac{4}{3} \right)^{N-1} 3^{m(N-1)}.
\]

Now (1.3) shows that we can apply Proposition 1 b) with \( \lambda = 3^{2N-1} \), which yields

\[
\alpha_N \geq (N-1) \left( 1 - \frac{\log 3}{2 \log 2} \right). \tag{4.2}
\]

We now provide an upper bound for \( \alpha_N \) based on the following lemma.

**Lemma 3:** For all \( m, N \in \mathbb{N} \),

\[
\int_0^{2\pi} \prod_{j=0}^{m-1} |\sin(2^j t)|^N dx \geq \frac{2\pi}{mN} 2^{mN/2} 3^{mN/2}.
\]

**Proof:** Since \( \sin^2(2x) = q(\sin^2 x) \), \( q(y) = 4y(1-y) \) and
Let \( q(j) = \frac{1}{4}, \) we have that \( \sin^2(2^{-j}x_0) = \frac{1}{4}, \) \( x_0 := 2\pi/3 \) for all \( j = 0, 1, 2, \cdots. \) Let \( k \) be the integer determined by
\[
x_0 \in [k2^{-m+2}\pi, (k+1)2^{-m+2}\pi].
\]
Then the function \( \sin(2^{-j-1}x) \) is positive and concave on the interval \( \left(k\pi, (k+1)\pi - \frac{\sqrt{3}}{2}\right) \) for all \( j = 0, \cdots, m-1. \)

Hence, if \( h \) denotes the continuous function that is zero outside the interval \( \left(k\pi, (k+1)\pi - \frac{\sqrt{3}}{2}\right), \) linear between the left endpoint and \( x_0 \) with \( h(x_0) = \sqrt{3}/2 \) and linear between \( x_0 \) and the right endpoint, we have
\[
\frac{\pi}{2N+1} \int_0^{2\pi} \sin(2^{-j}x) h(x)^m \, dx \geq h(x) \frac{\sqrt{3}}{2N+1}.
\]
Therefore,
\[
\int_0^{2\pi} \sin^2(2^{-j-1}x) \, dx \geq \int_0^{2\pi} h(x)^m \, dx = \frac{2\pi m^{-1}}{mN+1} 2^{-mN} \cos \left( \frac{\sqrt{3} - \frac{\sqrt{3}}{2}}{2^{-m+1}} \right).
\]

We now use this lemma to estimate the integral in (2.1). Note that, by (1.3) and Stirling’s formula, there is a positive constant \( C \) such that
\[
P_N(y) \geq \left( \frac{2N - 2}{N - 1} \right)^{yN-1} \geq C 2^{2N} \rho^{-1/2} yN, \quad 0 \leq y \leq 1.
\]
Then
\[
\prod_{j=0}^{m-1} |g_N(2^j t)| \geq C^{m/2} 2^{mN-m/4} \prod_{j=0}^{m-1} |\sin(2^{-j-1} t)|^N
\]
and
\[
\int_0^{2\pi} |\sin(2^{-j-1} t)| \prod_{j=0}^{m-1} |g_N(2^j t)| \, dt \geq C^{m/2} 2^{mN-m/4} \int_0^{2\pi} |\sin(2^{-j-1} t)|^N \, dt \geq C^{m/2} 2^{mN-m/4} \frac{2\pi}{m+1} \frac{2\pi}{mN+1} .
\]

This shows that we can take \( \lambda = \lambda_N \) where
\[
\lambda_N = C_0^{-1/4} N^{3/2}, \quad C_0 = \sqrt{C}/4.
\]

We obtain
\[
\alpha_N \leq N - \log(2C_0) - \frac{1}{2} \log N + N^{1/2} \log 3, \quad (4.4)
\]

The inequalities (4.2) and (4.4) prove the following result.

**Theorem 1:** The sequence \( \alpha_N/N \) converges to \( 1 - (\log 3)/(2 \log 2) = 0.2075 \cdots \) as \( N \to \infty. \)

**REFERENCES**


The Characterization of Continuous, Four-Coeficient Scaling Functions and Wavelets

David Colella and Christopher Heil

**Abstract—** Four-coefficient dilation equations are examined and results converse to a theorem of Daubechies-Lagarias are given. These results complete the characterization of those four-coefficient dilation equations having a continuous solution.

**Index Terms—** Dilation equation, Hölder continuity, joint spectral radius, multiresolution analysis, scaling function, wavelet.

A wavelet basis for \( L^2(R) \) is an orthonormal basis \( \{2^{-n/2} \psi(2^{-n} x - m)\}_{n,m \in \mathbb{Z}} \) generated from a single function \( \psi, \) the wavelet. The classical example is the Haar system, where \( \psi = \chi_{[0,1]} - \chi_{[1,2)} \). Wavelet bases have many applications, e.g., image and speech processing. The variety of applications demands that wavelet bases having specific properties be available. It is, therefore, important to have means by which wavelets with desired properties can be constructed. One method is to solve a dilation equation or two-scale difference equation \( f(t) = \sum_{k=-\infty}^{\infty} c_k f(2t - k). \) A functional solution \( f \) to a dilation equation is called a scaling function.

The wavelet \( \psi \) can be realized as \( \psi(t) = \sum_{k=-\infty}^{\infty} c_k f(2t - k) \) whenever the scaling function \( f \) defines a multiresolution analysis, cf. Example 1. Recently, numerous papers have addressed the issue of constructing scaling functions, e.g., the various subdivision schemes in [1]–[7]. The purpose of this correspondence is to characterize four-coefficient dilation equations having continuous solutions. One part of this characterization (Theorem 1) was pro-