$\therefore p'=[A\lambda+H\mu+G\nu-m(A+H+G)]/\triangle$.

$q'=[H\lambda+B\mu+F\nu-m(H+B+F)]/\triangle$.

$t'=[G\lambda+F\mu+C\nu-m(G+F+C)]/\triangle$.

(p, q, t), (p', q', t') are the tangential co-ordinates of the join of the poles.

Let A', B', C' be the angles of the triangle of reference. The center is the pole of the line at infinity $\alpha\sin A'+\beta\sin B'+\gamma\sin C'=0$. The tangential co-ordinates of the center are obtained by substituting $\sin A'$, $\sin B'$, $\sin C'$ for λ, μ, ν in p, q, t and are

$$S_1=(A\sin A'+H\sin B'+G\sin C')/\triangle,$$

$$S_2=(H\sin A'+B\sin B'+F\sin C')/\triangle,$$

$$S_3=(G\sin A'+F\sin B'+C\sin C')/\triangle.$$

$\therefore$ The tangential equation of the center is $\lambda S_1+\mu S_2+\nu S_3=0$.

Write $a+ka'$ for a, $b+kb'$ for b, $c+kc'$ for c, $f+kf'$ for f, $g+kg'$ for g, $h+kh'$ for h in $A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0$.

Then the tangential equation of the four points of intersection of S and S' is $S+k\Phi+k^2S'=0$ where k is undetermined, and

$$\Phi=(bc'+b'c-2ff')\lambda^2+(ca'+c'a-2gg')\mu^2+(ab'+a'b-2hh')\nu^2$$
$$+2(gh'+g'h-af'-a'f)\mu\nu+2(hf'+h'f-bg'-b'g)\pi\lambda$$
$$+2(fg'+f'g-ch'-c'h)\lambda\mu.$$

The condition for equal roots for k is $\Phi^2=4SS'$, which is the equation of the four points of intersection.

186. Proposed by J. R. HITT; Professor of Mathematics, Coronal Institute, San Marcos, Texas.

If two sides of a triangle and its in-circle be given in position, tne envelope of its circumcircle is a circle (*Mannheim*). [From Casey's *Sequel to Euclid.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let vertex A be origin, sides b, c the axes. Then $x^2+2xy\cos A+y^2-bx-cy=0$ is the equation to the circumcircle. Let this equation be written

$$D-bx-cy=0\ldots.(1).$$

Since the sides b, c and the inscribed circle are fixed in position, the tangents from A to the in-circle are constant.

$\therefore b+c-a=$ a constant$=m\ldots.(2)$.

$a=\sqrt{(b^2+c^2-2bc\cos A)}$. This in (2) gives after reduction,

$$m^2+2bc(1+\cos A)-2m(b+c)=0\ldots.(3).$$

c from (1) in (3) gives

$$2b^2x(1+\cos A)+2b[my-mx-D(1+\cos A)]+2Dm-m^2y=0.$$

The condition for equal roots of b is

$$2x(1+\cos A)(2Dm-m^2y)=[my-mx-D(1+\cos A)]^2$$

$$\text{or } [D(1+\cos A)-m(x+y)]^2=2m^2xy(1-\cos A)=4m^2xy\sin^2\tfrac{1}{2}A.$$

$$\therefore D(1+\cos A)-m(x+y)\pm 2m\surd(xy)\sin\tfrac{1}{2}A=0.$$

$$\therefore x^2+2xy\cos A+y^2-\frac{m}{2\cos^2\frac{1}{2}A}[x+y\pm 2(xy)\sin\tfrac{1}{2}A]=0.$$

$$\therefore x^2+2xy+y^2-4xy\sin^2\tfrac{1}{2}A-\frac{m}{2\cos^2\frac{1}{2}A}[x+y\pm 2\surd(xy)\sin\tfrac{1}{2}A]=0.$$

$$\therefore [x\pm 2\surd(xy)\sin\tfrac{1}{2}A+y][x\mp 2\surd(xy)\sin\tfrac{1}{2}A+y-\frac{m}{2\cos^2\frac{1}{2}A}]=0.$$

$$\therefore x\mp 2\surd(xy)\sin\tfrac{1}{2}A+y-\frac{m}{2\cos^2\frac{1}{2}A}=0,$$

$$\text{or } x^2+2xy\cos A+y^2+[m-4m(x+y)\cos^2\tfrac{1}{2}A]/4\cos^4\tfrac{1}{2}A=0.$$

This is the circle.

CALCULUS.

144. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

Find the volume of the sphere, $x^2+y^2+z^2=2az$, (a) within the paraboloid $z=Ax^2+By^2$; (b) within the cone $z^2=Ax^2+By^2$.

Solution by the PROPOSER.

$x^2+y^2+z^2=2az....(1)$, $z=Ax^2+By^2....(2)$, $z^2=Ax^2+By^2....(3)$.

From (1), $z=a\pm\surd\{a^2-x^2-y^2\}=a\pm\surd\{a^2-r^2\}$.

From $v=\int\int zrdrd\theta$ we get

$$v=4\int_0^{\frac{1}{2}\pi}\int_0^R\surd(a^2-r^2)d\theta rdr=\tfrac{4}{3}\int_0^{\frac{1}{2}\pi}]a^3-(a^2-r^2)^{\frac{3}{2}}]d\theta.$$

(a). From (1) and (2),

$$x^2+y^2+(Ax^2+By^2)^2=2a(Ax^2+By^2).$$

$$r^2+r^4(A\cos^2\theta+B\sin^2\theta)^2=2ar^2(A\cos^2\theta+B\sin^2\theta).$$